

# A perturbative approach to the reconstruction of the eigenvalue spectrum of a normal covariance matrix from a spherically truncated counterpart

Filippo Palombi\* and Simona Toti

*Istituto Nazionale di Statistica*  
Via Cesare Balbo 16, 00184 Rome – Italy

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## Abstract

In this paper we propose a perturbative method for the reconstruction of the covariance matrix of a multinormal distribution, under the assumption that the only available information amounts to the covariance matrix of a spherically truncated counterpart of the same distribution. We expand the relevant equations up to the fourth perturbative order and discuss the analytic properties of the first few perturbative terms. We finally compare the proposed approach with an exact iterative algorithm (presented in ref. [1]) in the hypothesis that the spherically truncated covariance matrix is estimated from samples of various sizes.

## 1 Introduction

In a recent paper [1], we have studied how the covariance matrix  $(\mathfrak{S}_{\mathcal{B}})_{ij} = \text{cov}(X_i, X_j | X \in \mathcal{B}_v(\rho))$  of a multinormal random vector  $X = \{X_k\}_{k=1}^v \sim \mathcal{N}_v(0, \Sigma)$  in  $v \geq 1$  dimensions, conditioned to a centered spherical domain  $\mathcal{B}_v(\rho) = \{x \in \mathbb{R}^v : x^\top x < \rho\}$ , relates to the unconditioned covariance matrix  $\Sigma_{ij} = \text{cov}(X_i, X_j)$ . Thanks to the symmetries of the geometrical set-up,  $\mathfrak{S}_{\mathcal{B}}$  and  $\Sigma$  can be shown to commute. Moreover, if we denote respectively by  $\mu = \{\mu_k\}_{k=1}^v$  and  $\lambda = \{\lambda_k\}_{k=1}^v$  the eigenvalue spectra of  $\mathfrak{S}_{\mathcal{B}}$  and  $\Sigma$ , then

$$\mu_k = \lambda_k \frac{\alpha_k(\rho; \lambda)}{\alpha(\rho; \lambda)}, \quad k = 1, \dots, v, \quad (1.1)$$

with  $\alpha$  and  $\alpha_k$  belonging to the class of Gaussian integrals

$$\alpha_{k\ell m \dots}(\rho; \lambda) = \int_{\mathcal{B}_v(\rho)} d^v x \frac{x_k^2}{\lambda_k} \frac{x_\ell^2}{\lambda_\ell} \frac{x_m^2}{\lambda_m} \dots \prod_{j=1}^v \delta(x_j, \lambda_j), \quad \delta(y, \eta) = \frac{e^{-y^2/(2\eta)}}{(2\pi\eta)^{1/2}}. \quad (1.2)$$

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\*Corresponding author. e-mail: [filippo.palombi@istat.it](mailto:filippo.palombi@istat.it)

Since  $\mathfrak{S}_{\mathcal{B}}$  and  $\Sigma$  are simultaneously diagonalizable, we can assume  $\Sigma = \text{diag}(\lambda)$  with no loss of generality. Reconstructing  $\Sigma$  from  $\mathfrak{S}_{\mathcal{B}}$  means solving eq. (1.1) with respect to  $\lambda$  under the assumption that  $\rho$  and  $\mu$  are given. This is only possible provided  $\mu$  fulfills specific algebraic constraints (see sect. 2 of [1]). Anyway, a closed-form solution is out of reach, due to the non-linear character of the problem. For this reason, we have proposed in [1] a numerical technique based on a fixed point iteration, whose convergence mechanism is related to some conjectured correlation inequalities within  $\mathcal{B}_v(\rho)$ .

In this paper, we approach eq. (1.1) from a different perspective. We move from the observation that a simplified set-up occurs when the eigenvalue spectra are fully degenerate, a case which has been first considered by Tallis [2]. If  $\mu_1 = \dots = \mu_v \equiv \tilde{\mu}$ , by symmetry it follows  $\lambda_1 = \dots = \lambda_v \equiv \tilde{\lambda}$ . The converse holds true as well. Eq. (1.1) reduces in this limit to

$$\tilde{\mu} = \tilde{\lambda} \frac{F_{v+2}}{F_v} \left( \frac{\rho}{\tilde{\lambda}} \right) \equiv \mathcal{T}_{\rho}(\tilde{\lambda}), \quad (1.3)$$

with  $F_v(x)$  denoting the *c.d.f.* of a  $\chi^2$ -variable with  $v$  degrees of freedom<sup>1</sup>. It can be readily checked that the function  $\mathcal{T}_{\rho}(\tilde{\lambda})$  is monotonic increasing in  $\tilde{\lambda}$ . In addition, we have

$$i) \lim_{\tilde{\lambda} \rightarrow 0} \mathcal{T}_{\rho}(\tilde{\lambda}) = 0, \quad ii) \lim_{\tilde{\lambda} \rightarrow \infty} \mathcal{T}_{\rho}(\tilde{\lambda}) = \frac{\rho}{v+2}, \quad (1.4)$$

whence we recognize that eq. (1.3) can be numerically inverted (by any root-finding algorithm) provided  $0 < \tilde{\mu} < \rho/(v+2)$ .

Now, eq. (1.3) can be thought of as the lowest order approximation of a perturbative expansion of eq. (1.1) around the point  $\lambda_{\text{T}} = \{\tilde{\lambda}, \dots, \tilde{\lambda}\}$ . If the condition number of  $\Sigma$  is not extremely large, such an expansion is expected to quickly converge, so that a few perturbative corrections to  $\lambda_{\text{T}}$  should be sufficient to guarantee a good level of approximation.

In critical applications, requiring reconstructions of the covariance matrix for several values of  $\rho$  or  $\mu$ , it could be important to access fast yet approximate solutions, such as perturbation theory provides, rather than slow yet exact ones. Indeed, the convergence of the fixed point iteration has been shown to slow down as  $\rho \rightarrow 0$  or  $v \rightarrow \infty$ , the rate of the slowing down being polynomial in the former limit and exponential in the latter. Thus, in all situations where  $\rho \ll \min_k \{\lambda_k\}$  or  $v \gg 1$ , the use of the fixed point algorithm could be unfavorable.

Another advantage of the perturbative approach turns up when  $\mathfrak{S}_{\mathcal{B}}$  is not known exactly, but instead it comes along as the result of a multivariate Gaussian sampling from a spherically truncated population of finite size. In that case, we shall see that the statistical fluctuations of the higher components of  $\mu$  are amplified by eq. (1.1) as a consequence of the non-linearity of the problem, sometimes resulting in unacceptable variances for the higher components of  $\lambda$ . In the framework of perturbation theory, non-linearity arises systematically on increasing the order of the approximation, since each perturbative correction depends non-linearly upon the previous ones. Therefore, by stopping the expansion at different orders, we have the possibility to define a class of statistical estimators of  $\lambda$ , each characterized by its own bias and variance. The variance of the upper (lower) half of the spectrum increases (decreases) along with the perturbative order, whereas the bias always decreases. In all applications where the the upper part of the spectrum counts, it

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<sup>1</sup> with abuse of notation we shall often write  $\frac{F_{v+2}}{F_v}(\frac{\rho}{\tilde{\lambda}})$  in place of  $F_{v+2}(\rho/\tilde{\lambda})/F_v(\rho/\tilde{\lambda})$ .

is therefore possible — at least in principle — to optimize the choice of the perturbative estimator according to one's needs.

Aim of the present paper is to carry out a theoretical study of the perturbative expansion of eq. (1.1) up to the fourth order and to discuss some analytic aspects of it. Here is a plan of the paper. In sect. 2 we derive some preliminary results concerning the integrals  $\alpha$  and  $\alpha_k$ , which are necessary for a systematic implementation of the perturbative strategy to all orders. In sect. 3 we work out the expansion by means of paper-and-pencil calculations and MAPLE<sup>TM</sup> programs (given in the appendix). Sect. 4 is devoted to discussing some analytic properties of the first few perturbative terms, when  $\tilde{\mu}$  is chosen to be the average of the truncated eigenvalues. Finally, in sect. 5 we simulate in sample space the statistical properties of the perturbative estimators of  $\lambda$  and the iterative one for a specific choice of the covariance matrix. Conclusions are drawn in sect. 6.

## 2 Building blocks in Tallis' limit

As widely known, perturbation theory is an expansion technique around a reference solution, which is assumed to be either calculable or easily computable. Its mathematical structure develops from building blocks which are themselves entirely defined in terms of the reference solution. When it comes to perturbing eq. (1.1) around  $\lambda_T$ , the elementary objects we need to focus on are the Gaussian integrals  $\alpha_{k\ell m\dots}$  and their partial derivatives in the limit  $\lambda \rightarrow \lambda_T$ , which we shall also refer to in the sequel as Tallis' limit.

To begin with, let us set-up the notation. Throughout the paper we shall keep on representing a fully degenerate spectrum by  $\lambda_T = \{\tilde{\lambda}, \dots, \tilde{\lambda}\}$ . Outside Tallis' limit an appropriate index reshuffling allows us to assume with no loss of generality the ordering  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_v$  (which is induced by  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_v$ , as asserted by Proposition 2.3 of [1]). We shall denote a generic Gaussian integral with  $n$  (not necessarily distinct) indices  $\{i_1, \dots, i_n\} \equiv \mathcal{I}$  either by the notation  $\alpha_{i_1\dots i_n}(\rho; \lambda)$  introduced in sect. 1 or by the *compact* notation  $\alpha_{1:m_1\dots v:m_v}(\rho; \lambda)$ , where subscript colons are meant to separate each directional index from the multiplicity it has in  $\mathcal{I}$ . By the same token, we shall denote the  $n^{\text{th}}$  order derivative operator with respect to the variances  $\lambda_{i_1} \dots, \lambda_{i_n}$  either by the *standard* symbol  $\partial_{i_1\dots i_n} = \partial^n / (\partial \lambda_{i_1} \dots \partial \lambda_{i_n})$  or by its *compact* version  $\partial_{1:m_1\dots v:m_v} = \partial^{m_1+\dots+m_v} / (\partial \lambda_1)^{m_1} \dots (\partial \lambda_v)^{m_v}$ . Of course, the multiplicity set  $\mathcal{M}_{\mathcal{I}} = \{m_1, \dots, m_v\}$  is unambiguously associated to  $\mathcal{I}$ . For consistency reasons it must fulfill  $\sum_{k=1}^v m_k = n$ . In case of vanishing multiplicities we shall drop all the corresponding indices. For instance, we shall write  $\alpha_{k:m_k}(\rho; \lambda)$  in place of the rather pedantic  $\alpha_{1:0\dots k:m_k\dots v:0}(\rho; \lambda)$  as well as  $\partial_{k:m_k}$  in place of  $\partial_{1:0\dots k:m_k\dots v:0}$ . Last but not least, we refer the reader to [1] for definitions and properties concerning the truncation operator  $\tau_\rho$  and its inverse  $\tau_\rho^{-1}$ .

Having said that, we start our investigation of Tallis' limit with a simple proposition, which is just meant to review the findings of [2]:

**Proposition 2.1.** *For  $m_1 \geq 0, \dots, m_v \geq 0$ , we have*

$$\alpha_{1:m_1\dots v:m_v}(\rho; \lambda_T) = \Delta_{1:m_1\dots v:m_v} F_{v+2(m_1+\dots+m_v)} \left( \frac{\rho}{\tilde{\lambda}} \right), \quad (2.1)$$

with

$$\Delta_{1:m_1\dots v:m_v} = \prod_{j=1}^v (2m_j - 1)!! . \quad (2.2)$$

*Proof.* In order to derive eq. (2.1), we simply need to represent  $\alpha_{1:m_1\dots v:m_v}(\rho; \lambda_T)$  in spherical coordinates,

$$\begin{aligned} x_1 &= r f_1(\theta_1, \dots, \theta_{v-1}) , \\ &\vdots \\ x_v &= r f_v(\theta_1, \dots, \theta_{v-1}) . \end{aligned} \quad (2.3)$$

Recall that  $\sum_k f_k^2 = 1$  and  $d^v x = r^{v-1} dr d\Omega$ , with  $d\Omega$  embodying the angular part of the Jacobian of eq. (2.3) and the differentials of the angles  $\theta_1, \dots, \theta_{v-1}$ . A few plain algebraic passages yield

$$\alpha_{1:m_1\dots v:m_v}(\rho; \lambda_T) = \Delta_{1:m_1\dots v:m_v} F_{v+2(m_1+\dots+m_v)} \left( \frac{\rho}{\tilde{\lambda}} \right) , \quad (2.4)$$

with the proportionality coefficient  $\Delta_{1:m_1\dots v:m_v}$  being independent of  $\rho$  or  $\tilde{\lambda}$ . In order to fix it, we observe that  $\alpha_{1:m_1\dots v:m_v}(\rho; \lambda_T)$  factorizes in  $v$  one-dimensional integrals as  $\rho \rightarrow \infty$ , corresponding to unconditioned univariate Gaussian moments of orders  $2m_1, \dots, 2m_v$ , normalized respectively by powers  $\tilde{\lambda}^{m_1}, \dots, \tilde{\lambda}^{m_v}$  of the common variance. Hence, we infer  $\Delta_{1:m_1\dots v:m_v} = \prod_{k=1}^v (2m_k - 1)!!$ .  $\square$

The  $\chi^2$ -c.d.f.  $F_{v+2(m_1+\dots+m_v)}(\rho/\tilde{\lambda})$  is intuitively interpreted as a correction factor incorporating all the effects of the volume conditioning. Obviously, it lessens the value of the integral except when  $\rho \rightarrow \infty$ , where it becomes ineffective.

Whenever the Gaussian integral is expressed in *standard* notation, eq. (2.1) can be only applied provided the index multiplicities are preliminarily counted. Nevertheless, it is reasonable to expect  $\Delta_{1:m_1\dots v:m_v}$  to be the *compact* representation of some not yet specified coefficient  $\Delta_{i_1\dots i_n}$ . By the same argument as above this is recognized to be

$$\Delta_{i_1\dots i_n} = \mathbb{E}[z_{i_1}^2 \dots z_{i_n}^2] , \quad \text{provided} \quad z_{i_k} \stackrel{\text{iid}}{\sim} \mathcal{N}(0, 1) , \quad k = 1, \dots, n . \quad (2.5)$$

Thanks to Isserlis' Theorem<sup>2</sup> [3],  $\Delta_{i_1\dots i_n}$  can be reduced to a sum of products of Kronecker symbols, namely

$$\Delta_{i_1\dots i_n} = \sum \prod \delta_{ij} , \quad (2.6)$$

where the sum extends over all distinct ways of partitioning the duplicated set  $\mathcal{I}^2 \equiv \{i_1, i_1, \dots, i_n, i_n\}$  into pairs. For later convenience it is worthwhile listing a few frequently recurring examples:

$$\Delta_{i_1} = 1 , \quad (2.7)$$

$$\Delta_{i_1 i_2} = 1 + 2\delta_{i_1 i_2} , \quad (2.8)$$

$$\Delta_{i_1 i_2 i_3} = 1 + 2(\delta_{i_1 i_2} + \delta_{i_1 i_3} + \delta_{i_2 i_3}) + 2(\delta_{i_1 i_2} \delta_{i_1 i_3} + \delta_{i_1 i_2} \delta_{i_2 i_3} + \delta_{i_1 i_3} \delta_{i_1 i_2}) + 2\delta_{i_1 i_2} \delta_{i_1 i_3} \delta_{i_2 i_3} , \quad (2.9)$$

$\vdots$

Eqs. (2.5)–(2.6) allow us to reformulate Proposition 2.1 according to

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<sup>2</sup>in the framework of mathematical physics the same result is universally known as *Wick's Theorem*.

**Corollary 2.1.** For  $n \geq 0$  and  $\{i_1, \dots, i_n\}$  a set of  $n$  (not necessarily distinct) indices, we have

$$\alpha_{i_1 \dots i_n}(\rho; \lambda_T) = \Delta_{i_1 \dots i_n} F_{v+2n} \left( \frac{\rho}{\lambda} \right), \quad (2.10)$$

with  $\Delta_{i_1 \dots i_n}$  as in eq. (2.6).

Now, in order to take arbitrarily high order derivatives of  $\alpha_{i_1 \dots i_n}(\rho; \lambda)$  with respect to the covariance eigenvalues, we can iterate the basic rule

$$(2\lambda_k \partial_k) \alpha_{i_1 \dots i_n}(\rho; \lambda) = \alpha_{i_1 \dots i_n k}(\rho; \lambda) - \left( 1 + 2 \sum_{j=1}^n \delta_{k i_j} \right) \alpha_{i_1 \dots i_n}(\rho; \lambda), \quad (2.11)$$

which follows by differentiation under the integral sign. Note that eq. (2.11) is not specifically related to spherical truncations, *i.e.* it is formally invariant under a reshaping of the truncation surface. It also shows that the differential operator  $2\lambda_k \partial_k$  behaves in a simpler manner than  $\partial_k$  when acting on  $\alpha_{i_1 \dots i_n}$ , in that it produces an integer linear combination of similar integrals. For this reason, the recurrence generated by  $\partial_{k:n}$  can be derived by first working out the action of the operator  $(2\lambda_k \partial_k)^n$  and then using

$$\partial_{k:n} = \frac{1}{\lambda_k^n} \sum_{j=1}^n \frac{(-1)^{n-j}}{2^j} \begin{bmatrix} n \\ j \end{bmatrix} (2\lambda_k \partial_k)^j, \quad (2.12)$$

where the symbols  $\begin{bmatrix} n \\ j \end{bmatrix}$  denote unsigned Stirling numbers of the first kind. Eq. (2.12) is a classic of combinatorial analysis. The reader is referred for instance to exercise 13, chap. 6 of ref. [4] for a proof of it. Iterated applications of the operator  $2\lambda_k \partial_k$  generate increasingly involved sums of Gaussian integrals, as asserted by

**Proposition 2.2.** For all  $j \geq 1$  and  $n \geq 0$ , we have

$$(2\lambda_k \partial_k)^j \alpha_{k:n} = \sum_{r=0}^j (-1)^{j-r} c_{jr}(n) \alpha_{k:(n+r)}, \quad (2.13)$$

with

$$c_{jr}(n) = \sum_{\ell_1=0}^r \sum_{\ell_2=0}^{\ell_1} \dots \sum_{\ell_{j-r}=0}^{\ell_{j-r-1}} \prod_{s=1}^{j-r} [2(n + \ell_s) + 1]. \quad (2.14)$$

*Proof.* The proof is by induction. We first note that  $c_{10}(n) = (2n+1)$  and  $c_{11}(n) = 1$ . Hence, for  $j=1$  eq. (2.13) agrees with eq. (2.11). Now, suppose that  $(2\lambda_k \partial_k)^j \alpha_{k:n}$  is well represented by eq. (2.13) with  $c_{jr}(n)$  as in eq. (2.14). Then,

$$\begin{aligned} (2\lambda_k \partial_k)^{j+1} \alpha_{k:n} &= \sum_{r=0}^j (-1)^{j-r} c_{jr}(n) (2\lambda_k \partial_k) \alpha_{k:(n+r)} \\ &= \sum_{r=0}^j (-1)^{j-r} c_{jr}(n) \{ \alpha_{k:(n+r+1)} - [2(n+r) + 1] \alpha_{k:(n+r)} \} \\ &= \sum_{r=0}^{j+1} (-1)^{j+1-r} \{ c_{j(r-1)}(n) + [2(n+r) + 1] c_{jr}(n) \} \alpha_{k:(n+r)}. \end{aligned} \quad (2.15)$$

Hence, the proof is complete if we are able to show that  $c_{jr}(n)$  fulfills the recurrence

$$c_{(j+1)r}(n) = c_{j(r-1)}(n) + [2(n+r) + 1]c_{jr}(n), \quad (2.16)$$

To this aim, we first calculate the second term on the *r.h.s.* as

$$\begin{aligned} [2(n+r) + 1]c_{jr}(n) &= \sum_{\ell_1=0}^r \sum_{\ell_2=0}^{\ell_1} \dots \sum_{\ell_{j-r}=0}^{\ell_{j-r-1}} [2(n+r) + 1] \prod_{s=1}^{j-r} [2(n + \ell_s) + 1] \\ &= \sum_{\ell_2=0}^r \sum_{\ell_3=0}^{\ell_2} \dots \sum_{\ell_{j+1-r}=0}^{\ell_{j-r}} [2(n+r) + 1] \prod_{s=2}^{j+1-r} [2(n + \ell_s) + 1] \\ &= \sum_{\ell_1=r}^r \sum_{\ell_2=0}^{\ell_1} \dots \sum_{\ell_{j+1-r}=0}^{\ell_{j-r}} \prod_{s=1}^{j+1-r} [2(n + \ell_s) + 1], \end{aligned} \quad (2.17)$$

and then we add it to the first one, thus obtaining

$$\begin{aligned} c_{j(r-1)}(n) + [2(n+r) + 1]c_{jr}(n) &= \sum_{\ell_1=0}^{r-1} \sum_{\ell_2=0}^{\ell_1} \dots \sum_{\ell_{j+1-r}=0}^{\ell_{j-r}} \prod_{s=1}^{j+1-r} [2(n + \ell_s) + 1] \\ &+ \sum_{\ell_1=r}^r \sum_{\ell_2=0}^{\ell_1} \dots \sum_{\ell_{j+1-r}=0}^{\ell_{j-r}} \prod_{s=1}^{j+1-r} [2(n + \ell_s) + 1] = c_{(j+1)r}(n). \end{aligned} \quad (2.18)$$

□

Nested sums similar to eq. (2.14) are considered for instance in [5], where all the cases in study are reduced to closed-form expressions with the help of special numbers, such as binomial coefficients, Stirling numbers, center factorial numbers, *etc.* Eq. (2.14) looks a bit harder to manage, since the summand is a product of non-homogeneous functions of the sum variables, whence it is not clear whether the nested sum can be ultimately evaluated in closed-form. However, as far as we are concerned, a perfectly convenient representation of  $c_{jr}(n)$  is provided by

**Proposition 2.3.** *For all  $j \geq 1$ ,  $0 \leq r \leq j$  and  $n \geq 0$ , we have*

$$c_{jr}(n) = \sum_{t=r}^j (-2)^{j-t} \frac{[2(n+t) - 1]!!}{[2(n+r) - 1]!!} \left\{ \begin{matrix} j \\ t \end{matrix} \right\} \binom{t}{r}, \quad (2.19)$$

with the symbols  $\left\{ \begin{matrix} j \\ t \end{matrix} \right\}$  denoting Stirling numbers of the second kind.

*Proof.* Let us denote by  $d_{jr}(n)$  the *r.h.s.* of eq. (2.19). For  $j = 1$ , we have  $d_{10}(n) = 2n + 1$  and  $d_{11}(n) = 1$ , which equal respectively  $c_{10}(n)$  and  $c_{11}(n)$ . Thus, we just need to prove that  $d_{jr}(n)$  obeys eq. (2.16). To this end, it is sufficient to make use of the basic recursive formulae  $\left\{ \begin{matrix} n+1 \\ m \end{matrix} \right\} = m \left\{ \begin{matrix} n \\ m \end{matrix} \right\} + \left\{ \begin{matrix} n \\ m-1 \end{matrix} \right\}$  and  $\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}$ . We detail the algebra just for the sake of completeness. We start from

$$d_{(j+1)r}(n) = \sum_{t=r}^{j+1} (-2)^{j+1-t} \frac{[2(n+t) - 1]!!}{[2(n+r) - 1]!!} \left\{ \begin{matrix} j+1 \\ t \end{matrix} \right\} \binom{t}{r}$$

$$\begin{aligned}
&= \sum_{t=r}^j (-2)^{j-t} \frac{[2(n+t)-1]!!}{[2(n+r)-1]!!} (-2t) \left\{ \begin{matrix} j \\ t \end{matrix} \right\} \binom{t}{r} \\
&+ \sum_{t=r-1}^j (-2)^{j-t} \frac{[2(n+t)+1]!!}{[2(n+r)-1]!!} \left\{ \begin{matrix} j \\ t \end{matrix} \right\} \binom{t+1}{r} \\
&= (2n+1)d_{jr}(n) + \sum_{t=r-1}^j (-2)^{j-t} \frac{[2(n+t)+1]!!}{[2(n+r)-1]!!} \left\{ \begin{matrix} j \\ t \end{matrix} \right\} \binom{t}{r-1}. \tag{2.20}
\end{aligned}$$

Then, we add and subtract  $(2r)d_{jr}(n)$  to the *r.h.s.* of eq. (2.20). Hence,

$$\begin{aligned}
d_{(j+1)r}(n) &= [2(n+r)+1]d_{jr}(n) \\
&+ \sum_{t=r-1}^j (-2)^{j-t} \frac{[2(n+t)-1]!!}{[2(n+r)-1]!!} \left\{ \begin{matrix} j \\ t \end{matrix} \right\} \left[ [2(n+t)+1] \binom{t}{r-1} - 2r \binom{t}{r} \right]. \tag{2.21}
\end{aligned}$$

Since  $r \binom{t}{r} = (t-r+1) \binom{t}{r-1}$ , the second term on the *r.h.s.* is recognized to be  $d_{j(r-1)}(n)$ .  $\square$

In view of [5] it is no surprise that  $c_{jr}(n)$  can be represented in terms of Stirling numbers. In addition, the presence of terms such as  $\left\{ \begin{matrix} j \\ t \end{matrix} \right\}$  fits perfectly when combining eq. (2.12) with eq. (2.13). We recall indeed that Stirling numbers of the first and second kind fulfill the identity

$$\sum_{t=0}^{\max\{j,k\}} (-1)^{t-k} \left\{ \begin{matrix} t \\ j \end{matrix} \right\} \left[ \begin{matrix} k \\ t \end{matrix} \right] = \delta_{jk}. \tag{2.22}$$

We have thereby collected all the ingredients needed to prove the main result of this section, namely

**Proposition 2.4.** *For  $m, n \geq 0$ , let  $\mathcal{K} = \{k_1, \dots, k_m\}$  and  $\mathcal{I} = \{i_1, \dots, i_n\}$  denote sets of (not necessarily distinct) indices, i.e.  $1 \leq k_j \leq v$  and  $1 \leq i_j \leq v$ . Then,*

$$\partial_{k_1 \dots k_m} \alpha_{i_1 \dots i_n}(\rho; \lambda_T) = \frac{1}{2^m \tilde{\lambda}^m} \Delta_{k_1 \dots k_m i_1 \dots i_n} \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} F_{v+2(j+n)} \left( \frac{\rho}{\tilde{\lambda}} \right). \tag{2.23}$$

In particular, for  $n = 0, 1$  we have

$$\partial_{k_1 \dots k_m} \alpha(\rho; \lambda_T) = \frac{1}{2^m \tilde{\lambda}^m} \Delta_{k_1 \dots k_m} \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} F_{v+2j} \left( \frac{\rho}{\tilde{\lambda}} \right), \tag{2.24}$$

$$\partial_{k_1 \dots k_m} \alpha_i(\rho; \lambda_T) = \frac{1}{2^m \tilde{\lambda}^m} \Delta_{k_1 \dots k_m i} \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} F_{v+2(j+1)} \left( \frac{\rho}{\tilde{\lambda}} \right). \tag{2.25}$$

*Proof.* First of all, let  $\mathcal{M}_{\mathcal{K}} = \{m_1, \dots, m_v\}$  and  $\mathcal{M}_{\mathcal{I}} = \{n_1, \dots, n_v\}$  be the multiplicity sets associated respectively to  $\mathcal{K}$  and  $\mathcal{I}$ , such that  $\partial_{k_1 \dots k_m} = \partial_{1:m_1 \dots v:m_v}$ ,  $\alpha_{i_1 \dots i_n} = \alpha_{1:n_1 \dots v:n_v}$  and

$$\partial_{k_1 \dots k_m} \alpha_{i_1 \dots i_n}(\rho; \lambda_T) = \partial_{1:m_1 \dots v:m_v} \alpha_{1:n_1 \dots v:n_v}(\rho; \lambda_T). \tag{2.26}$$

Recall also that  $\mathcal{M}_{\mathcal{K}}$  and  $\mathcal{M}_{\mathcal{I}}$  fulfill  $\sum_{\ell=1}^v m_{\ell} = m$  and  $\sum_{\ell=1}^v n_{\ell} = n$  for consistency reasons. Using eq. (2.12)  $m$  times yields

$$\begin{aligned} \partial_{k_1 \dots k_m} \alpha_{i_1 \dots i_n}(\rho; \lambda_{\mathcal{T}}) &= \frac{1}{\tilde{\lambda}^m} \sum_{j_1=1}^{m_1} \frac{(-1)^{m_1-j_1}}{2^{j_1}} \begin{bmatrix} m_1 \\ j_1 \end{bmatrix} \dots \sum_{j_v=1}^{m_v} \frac{(-1)^{m_v-j_v}}{2^{j_v}} \begin{bmatrix} m_v \\ j_v \end{bmatrix} \\ &\cdot \left[ (2\lambda_1 \partial_1)^{j_1} \dots (2\lambda_v \partial_v)^{j_v} \alpha_{1:n_1 \dots v:n_v} \right] (\rho; \lambda_{\mathcal{T}}). \end{aligned} \quad (2.27)$$

Moreover, with the help of eqs. (2.13) and (2.1), eq. (2.27) reduces to

$$\begin{aligned} &\partial_{k_1 \dots k_m} \alpha_{i_1 \dots i_n}(\rho; \lambda_{\mathcal{T}}) \\ &= \frac{1}{\tilde{\lambda}^m} \sum_{j_1=1}^{m_1} \frac{1}{2^{j_1}} \begin{bmatrix} m_1 \\ j_1 \end{bmatrix} \sum_{q_1=0}^{j_1} (-1)^{m_1-q_1} \dots \sum_{j_v=1}^{m_v} \frac{1}{2^{j_v}} \begin{bmatrix} m_v \\ j_v \end{bmatrix} \sum_{q_v=0}^{j_v} (-1)^{m_v-q_v} \\ &\cdot [2(n_1 + q_1) - 1]!! \dots [2(n_v + q_v) - 1]!! \\ &\cdot c_{j_1 q_1}(n_1) \dots c_{j_v q_v}(n_v) \cdot F_{v+2(s+q_1+\dots+q_v)} \left( \frac{\rho}{\tilde{\lambda}} \right). \end{aligned} \quad (2.28)$$

As a next step, we evaluate the coefficients  $c_{j_1 q_1}(n_1), \dots, c_{j_v q_v}(n_v)$  in terms of the expressions obtained in Proposition 2.3. Then, we make use of eq. (2.22) to make all Stirling numbers disappear. We finally identify  $\prod_{j=1}^v [2(n_j + m_j) - 1]!! = \Delta_{1:(m_1+n_1) \dots v:(m_v+n_v)}$  and so arrive at

$$\begin{aligned} \partial_{k_1 \dots k_m} \alpha_{i_1 \dots i_n}(\rho; \lambda_{\mathcal{T}}) &= \frac{1}{2^m \tilde{\lambda}^m} \Delta_{1:(m_1+n_1) \dots v:(m_v+n_v)} \\ &\cdot \sum_{q_1=0}^{m_1} \dots \sum_{q_v=0}^{m_v} (-1)^{m-(q_1+\dots+q_v)} \binom{m_1}{q_1} \dots \binom{m_v}{q_v} F_{v+2(s+q_1+\dots+q_v)} \left( \frac{\rho}{\tilde{\lambda}} \right). \end{aligned} \quad (2.29)$$

Two additional moves are still needed to complete the proof. In first place, we notice that the multiplicity set associated to  $\mathcal{K} \cup \mathcal{I} = \{k_1, \dots, k_m, i_1, \dots, i_n\}$  is  $\mathcal{M}_{\mathcal{K} \cup \mathcal{I}} = \{m_1 + n_1, \dots, m_v + n_v\}$ . Therefore, owing to Isserlis' Theorem we can identify  $\Delta_{1:(n_1+m_1) \dots v:(n_v+m_v)} = \Delta_{k_1 \dots k_m i_1 \dots i_n}$ . In second place, we observe that  $(q_1 + \dots + q_v)$  ranges from 0 to  $m$  when  $0 \leq q_1 \leq m_1, \dots, 0 \leq q_v \leq m_v$ . Accordingly, we can recast the *r.h.s.* of eq. (2.29) to

$$\partial_{k_1 \dots k_m} \alpha_{i_1 \dots i_n}(\rho; \lambda_{\mathcal{T}}) = \frac{1}{2^m \tilde{\lambda}^m} \Delta_{k_1 \dots k_m i_1 \dots i_n} \sum_{j=0}^m (-1)^{m-j} e_j F_{v+2(s+j)} \left( \frac{\rho}{\tilde{\lambda}} \right), \quad (2.30)$$

with

$$e_j = \sum_{q_1=0}^{m_1} \dots \sum_{q_v=0}^{m_v} \binom{m_1}{q_1} \dots \binom{m_v}{q_v} \delta_{q_1+\dots+q_v, j}. \quad (2.31)$$

This multiple sum can be easily calculated by reiterating Vandermonde's convolution  $\sum_k \binom{r}{k} \binom{s}{p-k} = \binom{r+s}{p}$ , which finally yields  $e_j = \binom{m}{j}$ .  $\square$



### 3 Perturbative expansion

In order to invert eq. (1.1), perturbation theory prescribes that we interpret  $\mu$  and  $\lambda$  as smooth functions of a parameter  $\epsilon \in [0, 1]$ , such that  $\lambda(\epsilon = 0) = \lambda_T$  and  $\lambda(\epsilon = 1) = \tau_\rho^{-1} \cdot \mu$ . We must consider  $\epsilon$  as an auxiliary variable, allowing us to pass continuously from Tallis' limit to the ultimate solution we are looking for. Then, we are supposed to expand  $\mu(\epsilon)$  and  $\lambda(\epsilon)$  in power series of  $\epsilon$  around the point  $\epsilon = 0$ , namely

$$\lambda_k(\epsilon) = \tilde{\lambda} + \epsilon \lambda_k^{(1)} + \epsilon^2 \lambda_k^{(2)} + \dots, \quad (3.1)$$

$$\mu_k(\epsilon) = \tilde{\mu} + \epsilon \mu_k^{(1)} + \epsilon^2 \mu_k^{(2)} + \dots. \quad (3.2)$$

For later convenience we shall denote by  $R_k$  the integral ratio  $\alpha_k/\alpha$ . Since  $R_k$  depends smoothly upon  $\lambda$ , it can be analogously expanded in power series of  $\epsilon$ . Thus, eq. (1.1) reads

$$\tilde{\mu} + \epsilon \mu_k^{(1)} + \epsilon^2 \mu_k^{(2)} + \dots = \left( \tilde{\lambda} + \epsilon \lambda_k^{(1)} + \epsilon^2 \lambda_k^{(2)} + \dots \right) \cdot \left( \tilde{R} + \epsilon R_k^{(1)} + \epsilon^2 R_k^{(2)} + \dots \right), \quad (3.3)$$

with  $\tilde{R} = R_k(\rho; \lambda_T)$  and  $R_k^{(n)} = (n!)^{-1} d^n R_k / d\epsilon^n |_{\epsilon=0}$  for  $n = 1, 2, \dots$ . The idea underlying perturbation theory is that we treat separately terms belonging to different perturbative orders, that is to say we equal terms of the same order in  $\epsilon$  on both sides of eq. (3.3) and then solve one by one the algebraic equations obtained.

There are of course some caveats.

*i)* Since  $\mu_k$  is an input parameter, an implementation of the perturbative strategy is only possible provided we prescribe how it enters the Taylor coefficients of the function  $\mu_k(\epsilon)$ . In principle, the assignment can be decided in complete freedom. For instance, the choice which will be made in the sequel is to concentrate  $\mu_k$  on the lowest available Taylor coefficient, *i.e.*

$$\begin{cases} \mu_k^{(0)} = \tilde{\mu}, \\ \mu_k^{(1)} = \mu_k - \tilde{\mu} \equiv \delta\mu_k, \\ \mu_k^{(n)} = 0, \quad n = 2, 3, \dots \end{cases} \quad (3.4)$$

Alternatively, we could spread  $\mu_k$  over all Taylor coefficients, *e.g.* according to the prescription

$$\begin{cases} \mu_k^{(0)} = \tilde{\mu}, \\ \mu_k^{(n)} = \frac{1}{n!} \{\log[1 + (\mu_k - \tilde{\mu})]\}^n, \quad n = 1, 2, \dots \end{cases} \quad (3.5)$$

Both eqs. (3.4) and (3.5) comply with the requirement  $\mu_k(\epsilon = 1) = \mu_k$ . Nevertheless, it must be borne in mind that each legitimate splitting of  $\mu$  affects differently the convergence properties of the perturbative series of  $\lambda$  as well as the statistical properties of the truncated series, when  $\mu$  is turned into a random variable in sample space. This will be further investigated in sect. 5.

*ii)* The specific choice of  $\tilde{\mu}$  is rather arbitrary: as far as we are concerned with the feasibility of the perturbative expansion, the only requirement to fulfill is that eq. (1.3) be invertible, which is

guaranteed provided  $0 \leq \tilde{\mu} \leq \rho/(v+2)$ . A convenient choice, as we shall see in the sequel, is represented by

$$\tilde{\mu} = \frac{1}{v} \sum_{k=1}^v \mu_k \equiv \bar{\mu}. \quad (3.6)$$

Proving that  $\bar{\mu} \leq \rho/(v+2)$  whenever  $\mu \in \mathcal{D}(\tau_\rho^{-1})$  is not trivial. Such an upper bound relies indeed on Gaussian correlation inequalities similar to those conjectured in sect. 2 of [1]. Here we give a tentative proof. If  $\mu \in \mathcal{D}(\tau_\rho^{-1})$ , then it follows  $\mu_k = \lambda_k(\alpha_k/\alpha)$  for some  $\lambda \in \mathbb{R}^v$ . In order to establish an upper bound to  $\bar{\mu}$ , we have to inspect how this varies as a function of  $\lambda$ , so we look at the derivatives

$$\begin{aligned} \partial_k \bar{\mu} &= \frac{1}{v} \sum_{i=1}^v \partial_k \mu_i = \frac{1}{v} \left[ \partial_k \mu_k + \sum_{i \neq k} \partial_k \mu_i \right] \\ &= \frac{1}{v} \left[ \partial_k \left( \lambda_k \frac{\alpha_k}{\alpha} \right) + \sum_{i \neq k} \lambda_i \partial_k \left( \frac{\alpha_i}{\alpha} \right) \right] = \frac{1}{2v\lambda_k^2} \sum_{i=1}^v \text{cov}(X_i^2, X_k^2 | X \in \mathcal{B}_v(\rho)) , \end{aligned} \quad (3.7)$$

where the rightmost expression has been obtained with the help of eqs. (2.9)–(2.11) of ref. [1]. In that paper we conjectured<sup>3</sup>  $\text{cov}(X_i^2, X_k^2 | X \in \mathcal{B}_v(\rho)) \leq 0$  for  $k \neq i$ . Here, we invoke the additional conjecture

$$\text{var}(X_k^2 | X \in \mathcal{B}_v(\rho)) \geq \sum_{i \neq k} |\text{cov}(X_i^2, X_k^2 | X \in \mathcal{B}_v(\rho))| , \quad (3.8)$$

which we motivate by the intuitive observation (supported with no exception by extensive numerical tests) that the negative square correlations induced by conditioning  $X$  to  $\mathcal{B}_v(\rho)$  are extremely weak. From eqs. (3.7)–(3.8) it follows

$$\sum_{k=1}^v \mu_k \leq \lim_{(\lambda_1, \dots, \lambda_v) \rightarrow (\infty, \dots, \infty)} \sum_{k=1}^v \lambda_k \frac{\alpha_k}{\alpha}(\rho; \tilde{\lambda}) = \frac{\int_{\mathcal{B}_v(\rho)} d^v x \left( \sum_{k=1}^v x_k^2 \right)}{\int_{\mathcal{B}_v(\rho)} d^v x} = \frac{v\rho}{v+2} . \quad (3.9)$$

This estimate improves the one given in eq. (2.17) *i*) of [1] and consequently leads us to conjecture  $\mathcal{D}(\tau_\rho^{-1}) \subseteq H'_v(\rho) \subset H_v(\rho)$ , where  $H_v(\rho)$  has been defined in [1] and

$$H'_v(\rho) = \left\{ x \in \mathbb{R}_+^v : \quad x_k \leq \min \left\{ \frac{\rho}{3}, \frac{\rho}{v-k+1} \right\} \text{ and } \sum_{k=1}^v x_k \leq \frac{v\rho}{v+2} \right\} \quad (3.10)$$

is a tighter bounding region.

**iii)** Perturbation theory works only provided the  $O(\epsilon^n)$ –equations

$$\mathcal{E}_k^{(n)} \equiv \sum_{j=0}^n \lambda_k^{(n-j)} R_k^{(j)} - \mu_k^{(n)} = 0, \quad n = 0, 1, \dots \quad (3.11)$$

---

<sup>3</sup>this conjecture makes only sense under the assumption  $X \sim \mathcal{N}_v(0, \Sigma)$  with  $\Sigma = \text{diag}(\lambda)$ .

generated by collecting all the  $O(\epsilon^n)$ -terms from eq. (3.3), establish an algebraic relation among the Taylor coefficients of  $\lambda(\epsilon)$  which can be solved with respect to  $\lambda^{(n)}$ . This allows us to represent the latter as a function  $\lambda^{(n)}(\tilde{\lambda}, \lambda^{(1)}, \dots, \lambda^{(n-1)}, \mu^{(n)})$  of its lower order companions and  $\mu^{(n)}$ . If such property is confirmed, as we shall argue in a moment, then inverting eq. (1.1) perturbatively amounts to solving one by one in sequence the systems of equations  $\{\mathcal{E}_k^{(1)} = 0\}_{k=1}^v$ ,  $\{\mathcal{E}_k^{(2)} = 0\}_{k=1}^v$ , ... up to a predefined order  $n$ . Establishing the level of precision thus achieved is a complicated matter, as typical when facing perturbative expansions. From a qualitative point of view, the approximation is certainly recognized to be correct up to  $O(\epsilon^{n+1})$ -terms, which however is not an estimate of the truncation error.

With regard to point *iii*), we notice that the only contributions to  $\mathcal{E}_k^{(n)}$  holding terms which are proportional to  $\lambda^{(n)}$  are precisely  $\lambda_k^{(n)} \tilde{R}$  and  $\tilde{\lambda} R_k^{(n)}$ . All the other contributions, of the form  $\lambda_k^{(n-j)} R_k^{(j)}$  with  $j = 1, \dots, n-1$ , can only depend upon  $\tilde{\lambda}, \lambda^{(1)}, \dots, \lambda^{(n-1)}$ . Indeed,  $R_k$  depends upon  $\epsilon$  implicitly via  $\lambda(\epsilon)$ , thus its  $j^{\text{th}}$  order derivative with respect to  $\epsilon$  distributes progressively according to the chain rule of differentiation. When evaluating such derivative at  $\epsilon = 0$ , all terms proportional to strictly positive powers of  $\epsilon$  vanish. As a consequence, every surviving term must be proportional to a product of Taylor coefficients of  $\lambda(\epsilon)$ , each belonging to  $\{\tilde{\lambda}, \lambda^{(1)}, \dots, \lambda^{(j)}\}$ . In particular, when  $j = n$  an explicit calculation yields

$$\text{terms in } \lambda^{(n)} \text{ within } \mathcal{E}_k^{(n)} : \quad \lambda_k^{(n)} \tilde{R} + \tilde{\lambda} \sum_{j=1}^v \lambda_j^{(n)} \partial_j R_k(\rho; \lambda_T). \quad (3.12)$$

In order to evaluate the first order partial derivatives of  $R_k$ , we make use of Propositions 2.1 and 2.4,

$$\begin{aligned} \partial_j R_k(\rho; \lambda_T) &= \left[ \frac{\partial_j \alpha_k}{\alpha} - \frac{\alpha_k \partial_j \alpha}{\alpha^2} \right] (\rho; \lambda_T) \\ &= \frac{1}{2\tilde{\lambda}} \left[ (1 + 2\delta_{jk}) \frac{F_{v+4}}{F_v} - \frac{F_{v+2}^2}{F_v^2} - 2\delta_{jk} \frac{F_{v+2}}{F_v} \right] \left( \frac{\rho}{\tilde{\lambda}} \right), \end{aligned} \quad (3.13)$$

whence eq. (3.11) reduces to<sup>4</sup>

$$\sum_{j=1}^v \mathcal{J}_{kj} \lambda_j^{(n)} = \mathcal{G}_k^{(n)} \left( \tilde{\lambda}, \lambda^{(1)}, \dots, \lambda^{(n-1)}, \mu^{(n)} \right), \quad (3.14)$$

with

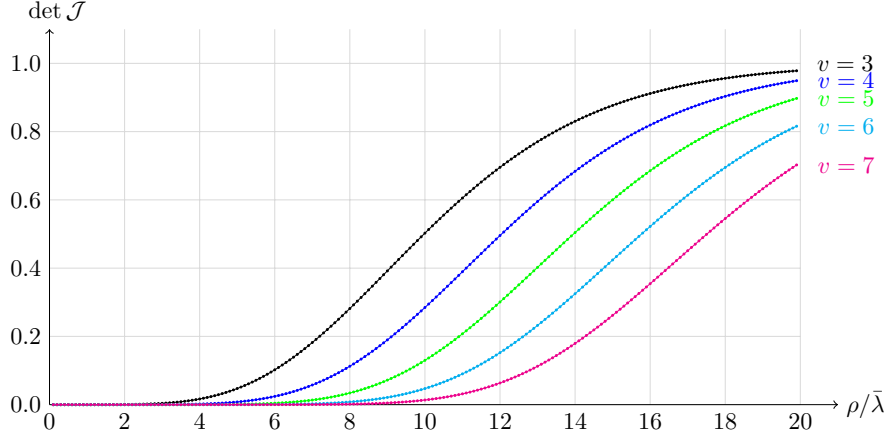
$$\mathcal{J}_{kj} = \frac{1}{2} \left[ (1 + 2\delta_{kj}) \frac{F_{v+4}}{F_v} - \frac{F_{v+2}^2}{F_v^2} \right]. \quad (3.15)$$

We have thereby obtained a system of linear equations with  $\lambda^{(n)}$  and  $\mathcal{G}^{(n)}$  representing respectively the variable and constant vectors. Moreover, the coefficient matrix  $\mathcal{J}$  is the Jacobian of the truncation operator  $\tau_\rho$  in Tallis' limit. Its determinant is given by

$$\det \mathcal{J} = \left( \frac{F_{v+4}}{F_v} \right)^{v-1} \left\{ \left( \frac{v}{2} + 1 \right) \frac{F_{v+4}}{F_v} - \frac{v}{2} \frac{F_{v+2}^2}{F_v^2} \right\} > \frac{2}{v+4} \left( \frac{F_{v+4}}{F_v} \right)^{v-1} \frac{F_{v+2}^2}{F_v^2}, \quad (3.16)$$

---

<sup>4</sup>from now on we shall drop the argument of the  $\chi^2$ -c.d.f.'s, which is always  $\rho/\tilde{\lambda}$ .



**Fig. 1** –  $\det \mathcal{J}$  vs.  $\rho/\tilde{\lambda}$  at  $v = 3, \dots, 7$ .

the lower bound on the *r.h.s.* following from the inequality  $F_{v+4}F_v/F_{v+2}^2 > (v+2)/(v+4)$ , first proved in [6]. Accordingly, we conclude that  $\mathcal{J}$  is non-singular for any finite value of  $\rho/\tilde{\lambda}$ , and therefore eq. (3.14) is unambiguously solved by  $\lambda^{(n)} = \mathcal{J}^{-1}\mathcal{G}^{(n)}$ . Note as well that  $\lim_{\rho/\tilde{\lambda} \rightarrow 0} \det \mathcal{J} = 0$ , so the invertibility of  $\mathcal{J}$  becomes critical at extremely small values of  $\rho/\tilde{\lambda}$ . By way of example, we show in Fig. 1 a plot of  $\det(\mathcal{J})$  vs.  $\rho/\tilde{\lambda}$  for  $v = 3, \dots, 7$ . Finally, the inverse of  $\mathcal{J}$  can be readily checked to be

$$(\mathcal{J}^{-1})_{jk} = \frac{F_v}{F_{v+4}} \frac{[(v+2)\delta_{jk} - 1] \frac{F_{v+4}}{F_v} - [v\delta_{jk} - 1] \frac{F_{v+2}^2}{F_v^2}}{(v+2) \frac{F_{v+4}}{F_v} - v \frac{F_{v+2}^2}{F_v^2}}. \quad (3.17)$$

Let us now come to the analytic structure of the known terms  $\mathcal{G}_k^{(n)}$ . We have just explained that owing to the chain rule of differentiation, every single contribution to  $\mathcal{E}_k^{(n)}$  (except for  $\mu_k^{(n)}$ ) holds a partial derivative  $\partial_{i_1} \dots \partial_{i_\ell} R_k(\rho; \lambda_T)$  with some indices  $i_1, \dots, i_\ell$ . On expanding this in terms of  $\alpha$  and  $\alpha_k$ , we produce ratios with numerators made of products of derivatives of  $\alpha$  and  $\alpha_k$  and denominators amounting to some power of  $\alpha$ . From the rules established by Propositions 2.1 and 2.4, it follows that  $\mathcal{G}_k^{(n)}$  can be represented in full generality as

$$\mathcal{G}_k^{(n)} = \mu_k^{(n)} + \tilde{\lambda}^{-n+1} \sum_{k_1=0}^{n+1} \sum_{k_2=0}^{k_1} \dots \sum_{k_{n+1}=0}^{k_n} c_{k;k_1\dots k_{n+1}}^{(n)} \frac{F_{v+2k_1} \dots F_{v+2k_{n+1}}}{F_v^{n+1}}, \quad (3.18)$$

$$K \equiv \sum_{j=1}^{n+1} k_j,$$

with the coefficient  $\tilde{\lambda}^{-n+1}$  factored out for later convenience. The subscript prescription  $K \leq n+1$  to the nested sum has to be understood as a restricting condition to the possible values taken by the sum variables  $k_1, \dots, k_{n+1}$ . For the sake of conciseness, we shall refer collectively to the ratios  $F_{v+2k_1} \dots F_{v+2k_{n+1}}/F_v^{n+1}$  as  $\chi^2$ -ratios and the coefficients  $c_{k;k_1\dots k_{n+1}}^{(n)}$  as *c*-coefs. We observe that

the *r.h.s.* of eq. (3.18) becomes increasingly populated for large values of  $n$ . In the few lowest order cases, it expands to

$$\mathcal{G}_k^{(1)} = \mu_k^{(1)} + c_{k;00}^{(1)} + c_{k;10}^{(1)} \frac{F_{v+2}}{F_v} + c_{k;11}^{(1)} \frac{F_{v+2}^2}{F_v^2} + c_{k;20}^{(1)} \frac{F_{v+4}}{F_v}, \quad (3.19)$$

$$\begin{aligned} \mathcal{G}_k^{(2)} = \mu_k^{(2)} + \tilde{\lambda}^{-1} \bigg( & c_{k;000}^{(2)} + c_{k;100}^{(2)} \frac{F_{v+2}}{F_v} + c_{k;110}^{(2)} \frac{F_{v+2}^2}{F_v^2} + c_{k;111}^{(2)} \frac{F_{v+2}^3}{F_v^3} \\ & + c_{k;200}^{(2)} \frac{F_{v+4}}{F_v} + c_{k;210}^{(2)} \frac{F_{v+4} F_{v+2}}{F_v^2} + c_{k;300}^{(2)} \frac{F_{v+6}}{F_v} \bigg), \end{aligned} \quad (3.20)$$

$\vdots$

Without conditioning the sum to  $K \leq n+1$ , the number of summands would be  $\binom{2n+2}{n+1}$  (*cf.* eq. (1) of [5]). Actually that number is much lower, yet  $\binom{2n+2}{n+1}$  can be taken as a (loose) upper bound to it. The intricacy of eq. (3.18) is only fake: upon adding separately the degrees of freedom of all the  $\chi^2$ -*c.d.f.* 's at numerator and denominator of each  $\chi^2$ -ratio and then subtracting the resulting numbers yields  $2K$ . Therefore, eq. (3.18) is just a formal way of representing a linear combination of  $\chi^2$ -ratios, where each  $\chi^2$ -*c.d.f.* has at least  $v$  degrees of freedom and the overall algebraic sum of degrees of freedom amounts to  $2K = 0, 2, \dots, 2(n+1)$  (with denominators contributing negatively). We stress once more that this analytic structure is a direct consequence of the chain rule of differentiation alongside with the results established in Propositions 2.1 and 2.4.

As for the  $c$ -coefs, they do not depend on  $\rho$  and can be only determined by direct calculation. In spite of this, their dependence upon the Taylor coefficients of  $\lambda(\epsilon)$  displays a well defined analytic structure. In order to disclose it, we must rely upon the notions of physical and perturbative dimensions.

**i)** First of all, we assume that  $\lambda$  has the physical dimension of length ( $L$ ), for which we adopt the notation  $[\lambda]_L = L^1$ . If we also assume  $[\epsilon]_L = L^0$ , then it follows  $[\tilde{\lambda}]_L = [\lambda^{(1)}]_L = \dots = [\lambda^{(n)}]_L = L^1$ . Similarly, we have  $[\mu]_L = L^1$  and  $[R_k]_L = L^0$ . Since  $[\mathcal{J}]_L = L^0$ , from eq. (3.14) it follows  $[\mathcal{G}^{(n)}]_L = L^1$ . Hence, eq. (3.18) leads us to conclude

$$[c_{k;k_1 \dots k_{n+1}}^{(n)}]_L = L^n. \quad (3.21)$$

As previously explained,  $c$ -coefs can only depend polynomially upon the Taylor coefficients of  $\lambda(\epsilon)$ . Eq. (3.21) tells us that such polynomials must be linear combinations of monomials in  $\tilde{\lambda}$  and the directional components of  $\lambda^{(1)}, \dots, \lambda^{(n-1)}$ , each monomial having precisely degree  $n$ .

**ii)** We define the perturbative dimension of a single monomial as the sum of the perturbative orders of its factors. More precisely, we set  $[\tilde{\lambda}]_P = P^0$ ,  $[\lambda^{(1)}]_P = P^1$ ,  $\dots$ ,  $[\lambda^{(n)}]_P = P^n$ . Thus, for instance, we have  $[\tilde{\lambda}^2 (\lambda_i^{(2)})^3 (\lambda_j^{(1)})^2]_P = P^8$ . We remark that  $\tilde{\lambda}$  carries no perturbative dimension, but it does increase the physical dimension of the monomials. Since  $\mathcal{G}^{(n)}$  is the result of the expansion at  $O(\epsilon^n)$ , it follows

$$[c_{k;k_1 \dots k_{n+1}}^{(n)}]_P = P^n. \quad (3.22)$$

The same result applies clearly to each single contributing monomial.

iii) Several monomials within a given  $c$ -coef have the same perturbative structure and numerical prefactor and differ only by directional indices, *e.g.*  $3\tilde{\lambda}^2(\lambda_1^{(2)})^3(\lambda_2^{(1)})^2$  and  $3\tilde{\lambda}^2(\lambda_4^{(2)})^3(\lambda_5^{(1)})^2$ . This is a consequence of the index structure of  $\Delta_{i_1\dots i_\ell}$ : the products of Kronecker symbols contributing to the *r.h.s.* of eq. (2.6) contract the indices of the Taylor coefficients of  $\lambda(\epsilon)$  in all possible ways, thus generating an increasing number of new aggregate structures at each order of the expansion. For instance, monomials within  $c$ -coefs belonging to the lowest perturbative orders are grouped according to

$$O(\epsilon^2) : \quad \zeta_1 = \sum_{i=1}^v \lambda_i^{(1)}, \quad \zeta_{1:2} = \sum_{i=1}^v (\lambda_i^{(1)})^2; \quad (3.23)$$

$$O(\epsilon^3) : \quad \zeta_2 = \sum_{i=1}^v \lambda_i^{(2)}, \quad \zeta_{12} = \sum_{i=1}^v \lambda_i^{(1)} \lambda_i^{(2)}, \quad \zeta_{1:3} = \sum_{i=1}^v (\lambda_i^{(1)})^3; \quad (3.24)$$

$$O(\epsilon^4) : \quad \zeta_3 = \sum_{i=1}^v \lambda_i^{(3)}, \quad \zeta_{13} = \sum_{i=1}^v \lambda_i^{(1)} \lambda_i^{(3)}, \quad \zeta_{2:2} = \sum_{i=1}^v (\lambda_i^{(2)})^2, \\ \zeta_{1:2:2} = \sum_{i=1}^v (\lambda_i^{(1)})^2 \lambda_i^{(2)}, \quad \zeta_{1:4} = \sum_{i=1}^v (\lambda_i^{(1)})^4; \quad (3.25)$$

$\vdots$

In view of the above considerations, we conclude that all  $c$ -coefs at  $O(\epsilon^n)$  with  $n \geq 2$  can be represented in full generality as linear combinations of all possible products of perturbative structures under the constraints imposed by eqs. (3.21) and (3.22), *i.e.*

$$c_{k;k_1\dots k_{n+1}}^{(n)} = \sum_m \gamma_{k_1\dots k_{n+1}}^{(n,m)} \mathcal{O}_{k;m}^{(n)}, \quad (3.26)$$

with numerical prefactors  $\gamma_{k_1\dots k_{n+1}}^{(n,m)}$  and perturbative structures  $\mathcal{O}_{k;m}^{(n)}$  fulfilling  $[\mathcal{O}_{k;m}^{(n)}]_L = L^n$  and  $[\mathcal{O}_{k;m}^{(n)}]_P = P^n$ . For instance, we have

$$\mathcal{O}_{k;m}^{(2)} \in \{(\lambda_k^{(1)})^2, \lambda_k^{(1)} \zeta_1, \zeta_1^2, \zeta_{1:2}\}, \quad (3.27)$$

$$\mathcal{O}_{k;m}^{(3)} \in \{(\lambda_k^{(1)})^3, (\lambda_k^{(1)})^2 \zeta_1, \lambda_k^{(1)} \zeta_1^2, \zeta_1^3, \lambda_k^{(1)} \zeta_{1:2}, \zeta_{1:3}, \\ \zeta_1 \zeta_{1:2}, \tilde{\lambda} \lambda_k^{(1)} \lambda_k^{(2)}, \tilde{\lambda} \lambda_k^{(1)} \zeta_2, \tilde{\lambda} \zeta_{12}, \tilde{\lambda} \lambda_k^{(2)} \zeta_1, \tilde{\lambda} \zeta_1 \zeta_2\}, \quad (3.28)$$

$\vdots$

Before working out the expansion at a given order, one should write down a complete set of perturbative structures pertaining to that order, such as eqs. (3.27) and (3.28) illustrate for  $n = 2, 3$ . A preliminary identification of all suitable structures is indeed particularly useful in order to identify groups of terms when high order calculations are performed by means of a computer algebra system (CAS), as we shall see in sect. 3.3.

When calculating  $c_{k;k_1\dots k_{n+1}}^{(n)}$ , many of the coefficients  $\gamma_{k_1\dots k_{n+1}}^{(n,m)}$  are found to be zero. The non-vanishing ones are subject to

**Proposition 3.1.** For  $n \geq 2$  the coefficients  $\gamma_{k_1 \dots k_{n+1}}^{(n,m)}$  are constrained by

$$\sum_{k_1=0}^{n+1} \sum_{k_2=0}^{k_1} \dots \sum_{k_{n+1}=0}^{k_n} \gamma_{k_1 \dots k_{n+1}}^{(n,m)} = 0. \quad (3.29)$$

*Proof.* We first note that if  $\mu \in \mathcal{D}(\tau_\rho^{-1})$ , then  $\mu \in \mathcal{D}(\tau_{\rho'}^{-1}) \quad \forall \rho' > \rho$ . Therefore, it makes sense to consider eq. (3.14) as  $\rho \rightarrow \infty$  with  $\mu$  kept fixed. In particular, we have showed previously that  $\lim_{\rho \rightarrow \infty} \mathcal{J}_{kj} = \delta_{kj}$ . Moreover, as  $\rho \rightarrow \infty$  all the  $\chi^2$ -ratios tend to one, thus eq. (3.14) reduces to

$$\begin{aligned} \lambda_k^{(n)} &= \mu_k^{(n)} + \sum_{k_1=0}^{n+1} \sum_{k_2=0}^{k_1} \dots \sum_{k_{n+1}=0}^{k_n} c_{k; k_1 \dots k_{n+1}}^{(n)} = \mu_k^{(n)} + \sum_{k_1=0}^{n+1} \sum_{k_2=0}^{k_1} \dots \sum_{k_{n+1}=0}^{k_n} \sum_m \gamma_{k_1 \dots k_{n+1}}^{(n,m)} \mathcal{O}_{k;m}^{(n)} \\ &= \mu_k^{(n)} + \sum_m \mathcal{O}_{k;m}^{(n)}(\tilde{\lambda}, \lambda^{(1)}, \dots, \lambda^{(n-1)}) \left[ \sum_{k_1=0}^{n+1} \sum_{k_2=0}^{k_1} \dots \sum_{k_{n+1}=0}^{k_n} \gamma_{k_1 \dots k_{n+1}}^{(n,m)} \right] \end{aligned} \quad (3.30)$$

However, in the same limit  $\lambda_k \rightarrow \mu_k$ , which entails order by order  $\lambda_k^{(n)} \rightarrow \mu_k^{(n)}$ . Hence, we infer that the sum of perturbative structures on the *r.h.s.* of eq. (3.30) vanishes as  $\rho \rightarrow \infty$ . Since in general  $\mathcal{O}_{k;m}^{(n,m)}(\tilde{\mu}, \mu^{(1)}, \dots, \mu^{(n-1)}) \neq 0$ , we must conclude that eq. (3.29) holds true.  $\square$

Now, the first few orders of the perturbative expansion can be worked out easily with a little algebra. Doing the calculations is useful to get acquainted with the general structure examined so far. The lowest order has been discussed in sect. 1, so we shall concentrate on the perturbative corrections to it. As of now, we shall assume that  $\mu(\epsilon)$  is expanded according to eq. (3.4).

### 3.1 Perturbative expansion at $\mathcal{O}(\epsilon^1)$

Equations  $\{\mathcal{E}_k^{(1)} = 0\}_{k=1}^v$  have the explicit form

$$\delta\mu_k = \lambda_k^{(1)} \tilde{R} + \tilde{\lambda} R_k^{(1)}. \quad (3.31)$$

The only term we need to calculate is

$$R_k^{(1)} = \left. \frac{dR_k}{d\epsilon} \right|_{\epsilon=0} = \sum_{j=1}^v \lambda_j^{(1)} \partial_j R_k(\rho; \lambda_T). \quad (3.32)$$

Actually, we have calculated the first order partial derivatives of  $R_k$  in eq. (3.13). Thus, we have

$$\sum_{j=1}^v \mathcal{J}_{kj} \lambda_j^{(1)} = \delta\mu_k. \quad (3.33)$$

whence we infer  $\mathcal{G}_k^{(1)} = \delta\mu_k$ . Choosing  $\tilde{\mu} = \bar{\mu}$  yields an important simplification:

**Proposition 3.2.** *If  $\tilde{\mu} = \bar{\mu}$ , then  $\zeta_1 = 0$ .*

*Proof.* It is sufficient to add side by side all eqs. (3.33) for  $k = 1, \dots, v$  to get

$$\frac{\zeta_1}{2} \left[ (v+2) \frac{F_{v+4}}{F_v} - v \frac{F_{v+2}^2}{F_v^2} \right] = \sum_{k=1}^v \delta\mu_k = 0. \quad (3.34)$$

Since the quantity in square brackets is strictly positive, we must conclude that  $\zeta_1 = 0$ .  $\square$

As can be readily understood, annihilating  $\zeta_1$  results in a huge simplification of the higher order calculations. Indeed,  $\zeta_1$  figures in many perturbative structures contributing to  $\mathcal{G}^{(n)}$  for  $n \geq 2$ . For instance, the structure basis of eq. (3.27) is reduced to only two elements in place of four, while the one of eq. (3.28) is reduced to six elements in place of twelve.

### 3.2 Perturbative expansion at $O(\epsilon^2)$

The subleading correction  $\lambda^{(2)}$  is obtained from the equations

$$0 = \lambda_k^{(2)} \tilde{R} + \lambda_k^{(1)} R_k^{(1)} + \tilde{\lambda} R_k^{(2)}. \quad (3.35)$$

Most of the contributions to the three terms on the *r.h.s.* are calculated smoothly at this point. For instance, we have

$$\lambda_k^{(1)} R_k^{(1)} = \frac{1}{\tilde{\lambda}} \left[ 2(\lambda_k^{(1)}) + \lambda_k^{(1)} \zeta_1 \right] \frac{F_{v+4}}{F_v} - \frac{\lambda_k^{(1)} \zeta_1}{\tilde{\lambda}} \frac{F_{v+2}^2}{F_v^2} - \frac{(\lambda_k^{(1)})^2}{\tilde{\lambda}} \frac{F_{v+2}}{F_v}, \quad (3.36)$$

$$\lambda_k^{(2)} \tilde{R} + \tilde{\lambda} R_k^{(2)} = \frac{1}{2} \frac{d^2 R_k}{d\epsilon^2} \Big|_{\epsilon=0} = \sum_{j=1}^v \mathcal{J}_{kj} \lambda_j^{(2)} + \frac{\tilde{\lambda}}{2} \sum_{j_1 j_2=1}^v \lambda_{j_1}^{(1)} \lambda_{j_2}^{(1)} \partial_{j_1 j_2} R_k(\rho; \lambda_T). \quad (3.37)$$

To keep things general, we make no assumptions on  $\tilde{\mu}$  here. Accordingly, we keep all  $\chi^2$ -ratios proportional to powers of  $\zeta_1$ . The evaluation of the second derivatives  $\partial_{j_1 j_2} R_k(\rho; \lambda_T)$  requires a few pages of tedious algebraic work, which we cannot detail for obvious reasons. The upshot is given by

$$\begin{aligned} \sum_{j=1}^v \mathcal{J}_{kj} \lambda_j^{(2)} &= -\frac{1}{8\tilde{\lambda}} \left[ \zeta_1^2 + 2\zeta_{1:2} + 4\lambda_k^{(1)} \zeta_1 + 8(\lambda_k^{(1)})^2 \right] \frac{F_{v+6}}{F_v} \\ &+ \frac{1}{8\tilde{\lambda}} \left[ 3\zeta_1^2 + 2\zeta_{1:2} + 4\lambda_k^{(1)} \zeta_1 \right] \frac{F_{v+4} F_{v+2}}{F_v^2} - \frac{\zeta_1^2}{4\tilde{\lambda}} \frac{F_{v+2}^3}{F_v^3} \\ &+ \frac{1}{2\tilde{\lambda}} \left[ \zeta_{1:2} + 2(\lambda_k^{(1)})^2 \right] \frac{F_{v+4}}{F_v} - \frac{\zeta_{1:2}}{2\tilde{\lambda}} \frac{F_{v+2}^2}{F_v^2}. \end{aligned} \quad (3.38)$$

We notice that the four structures listed in eq. (3.27) all contribute to the  $c$ -coefs  $c_{k;k_1 k_2 k_3}^{(2)}$ .



	$\frac{F_{v+6}}{F_v}$	$\frac{F_{v+4}F_{v+2}}{F_v^2}$	$\frac{F_{v+2}^3}{F_v^3}$	$\frac{F_{v+4}}{F_v}$	$\frac{F_{v+2}^2}{F_v^2}$
$(\lambda_k^{(1)})^2$	-1	0	0	1	0
$\lambda_k^{(1)}\zeta_1$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\zeta_1^2$	$-\frac{1}{8}$	$\frac{3}{8}$	$-\frac{1}{4}$	0	0
$\zeta_{1:2}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	$-\frac{1}{2}$

**Table 1** – Coefficients  $\gamma_{k_1 k_2 k_3}^{(2,m)}$ .

In Table 1 we collect the coefficients  $\gamma_{k_1 k_2 k_3}^{(2,m)}$ . Instead of naming rows and columns respectively according to the values of  $m$  and the triples  $(k_1, k_2, k_3)$ , for the sake of readability we identify each table entry by the perturbative-structure and the  $\chi^2$ -ratio it refers to. This way of tabulating the coefficients becomes particularly informative at higher orders. We finally observe that adding the entries of each table row yields zero, in accordance with eq. (3.29).

### 3.3 Perturbative expansion at higher orders

Paper-and-pencil calculations become prohibitively expensive at higher orders. Fortunately, it is not difficult to work out the algebra with the assistance of a CAS. For the reader's convenience, in the appendix we attach prototype MAPLE<sup>TM</sup> procedures, which facilitate the task. The code is split into three blocks, which we shortly review.

The first code block (A.1) contains a procedure `Delta()`, which computes the coefficient  $\Delta_{i_1 \dots i_n}$ . The procedure argument is assumed to be a list of **nonnegint** items; alternatively the procedure returns unevaluated. The input list is first sorted in ascending order, then the multiplicity set is identified. The procedure computes the *r.h.s.* of eq. (2.2) and returns its numerical value.

The second code block (A.2) performs the algebraic work related to the perturbative expansion of eq. (1.1). Before submitting it to evaluation, the user is assumed to assign a **nonnegint** variable **v** representing the number of dimensions, and a **nonnegint** variable **n**  $\leq 4$  representing the highest perturbative order processed by the program. The code block starts with a pair of procedures, `DerAlpha()` and `DerAlphak()`, which encode respectively eqs. (2.24) and (2.25). Afterwards, it performs a Taylor expansion of the *r.h.s.* of eq. (1.1) up to  $O(\epsilon^n)$ . Taylor coefficients are stored within an **indexable** object **h[j,k]**, the indices **j** and **k** representing respectively the perturbative order and the physical direction. At this stage, **h[j,k]** holds a sum of potentially many terms. The summands contain derivatives of  $\alpha$  and  $\alpha_k$ , which are purely symbolic objects. Their evaluation requires sequences of prescriptions, stored within the variables **COA**, **COAk**, ..., **C4A**, **C4Ak**. Algebraic simplifications are performed in the last few lines, where partial results are stored within **indexable** objects **h00**, ..., **h10**, so as to allow for an offline oversight of the single steps.

The third code block (A.3) illustrates in a specific case a numerical technique which we have devised for the determination of the coefficients  $\gamma_{k_1 \dots k_{n+1}}^{(n,m)}$ . The code processes the coefficients

	$\frac{F_{v+8}}{F_v}$	$\frac{F_{v+6}F_{v+2}}{F_v^2}$	$\frac{F_{v+4}^2}{F_v^2}$	$\frac{F_{v+6}}{F_v}$	$\frac{F_{v+4}F_{v+2}}{F_v^2}$	$\frac{F_{v+4}}{F_v}$	$\frac{F_{v+2}^2}{F_v^2}$
$(\lambda_k^{(1)})^3$	-1	0	0	2	0	-1	0
$\lambda_k^{(1)}\zeta_{1:2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\zeta_{1:3}$	$-\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\tilde{\lambda}\lambda_k^{(1)}\lambda_k^{(2)}$	0	0	0	-2	0	2	0
$\tilde{\lambda}\lambda_k^{(1)}\zeta_2$	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0
$\tilde{\lambda}\zeta_{12}$	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1

**Table 2** – Coefficients  $\gamma_{k_1 k_2 k_3 k_4}^{(3,m)}$ . We assume in this case  $\zeta_1 = 0$ .

corresponding to  $n = 3$  and  $(k_1, k_2, k_3, k_4) = (3, 0, 0, 0)$ , *i.e.* those entering the  $c$ -coef multiplying the  $\chi^2$ -ratio  $F_{v+6}/F_v$ . It also assumes  $\tilde{\mu} = \bar{\mu}$ , which reduces the basis of perturbative structures to

$$\mathcal{O}_{k;m}^{(3)} \in \{(\lambda_k^{(1)})^3, \lambda_k^{(1)}\zeta_{1:2}, \zeta_{1:3}, \tilde{\lambda}\lambda_k^{(1)}\lambda_k^{(2)}, \tilde{\lambda}\lambda_k^{(1)}\zeta_2, \tilde{\lambda}\zeta_{12}\}. \quad (3.39)$$

The algebraic sum pointed to by `h10[3,k]` at the end of the second code block has no knowledge of these structures. In order to identify them within `h10[3,k]`, we need to group terms properly. Instead of proceeding at an algebraic level, which turns out to be computationally demanding, we adopt a numerical approach, based on the use of eq. (3.26) as a square linear system fulfilled by the coefficients  $\gamma_{k_1 \dots k_{n+1}}^{(n,m)}$ . Having subtracted from `h10[3,k]` all contributions figuring in eq. (3.12), we extract from it all terms proportional to  $F_{v+6}/F_v$ , whose sum amounts to  $-c_{k;3000}^{(3)}$ . Then, for each  $k$  we assign  $\lambda^{(1)}$  and  $\lambda^{(2)}$  random values (chosen so that  $\zeta_1 = 0$ ), from which we compute  $\mathcal{O}_{k;1}^{(3)}, \dots, \mathcal{O}_{k;6}^{(3)}$  and  $c_{k;3000}^{(3)}$ . The random matrix  $\mathcal{O}_{k;m}^{(3)}$  thus obtained is non-singular, so eq. (3.12) can be solved with respect to  $\gamma_{3000}^{(3,m)}$ . The solution must be independent of the random numbers extracted. This represents a strong hint of goodness of our determination, but a real check consists of an algebraic comparison between the reconstructed coefficient  $c_{k;3000}^{(3)}$  and the one extracted from `h10[3,k]`. In Tables 2 and 3 we report the coefficients  $\gamma_{k_1 k_2 k_3 k_4}^{(3,m)}$  and  $\gamma_{k_1 k_2 k_3 k_4 k_5}^{(4,m)}$  under the assumption  $\zeta_1 = 0$ .

## 4 Properties of the first few perturbative coefficients

So far we have focused on formal aspects of the perturbative expansion with the aim of proving its theoretical and computational feasibility. When it comes to establish the level of accuracy reached by approximating the reconstruction with a finite number of contributions, we are soon led to investigate the analytic properties of the perturbative coefficients of  $\lambda$ .

	$\frac{F_{v+10}}{F_v}$	$\frac{F_{v+8}F_{v+2}}{F_v^2}$	$\frac{F_{v+6}F_{v+4}}{F_v^2}$	$\frac{F_{v+4}^2F_{v+2}}{F_v^3}$	$\frac{F_{v+8}}{F_v}$	$\frac{F_{v+6}F_{v+2}}{F_v^2}$	$\frac{F_{v+4}^2}{F_v^2}$	$\frac{F_{v+4}F_{v+2}^2}{F_v^3}$	$\frac{F_{v+6}}{F_v}$	$\frac{F_{v+4}F_{v+2}}{F_v^2}$	$\frac{F_{v+2}^3}{F_v^3}$	$\frac{F_{v+4}}{F_v}$	$\frac{F_{v+2}^2}{F_v^2}$
$(\lambda_k^{(1)})^4$	-1	0	0	0	3	0	0	0	-3	0	0	1	0
$(\lambda_k^{(1)})^2\zeta_{1:2}$	-1/4	0	1/4	0	3/4	-1/2	-1/4	0	-1/2	1/2	0	0	0
$\lambda_k^{(1)}\zeta_{1:3}$	-1/6	0	1/6	0	1/2	0	-1/2	0	-1/2	1/2	0	0	0
$\zeta_{1:2}^2$	-1/32	1/32	1/16	-1/16	1/8	-1/4	-1/8	1/4	-1/8	3/8	-1/4	0	0
$\zeta_{1:4}$	-1/8	1/8	0	0	1/2	-1/2	0	0	-3/4	3/4	0	1/2	-1/2
$\tilde{\lambda}(\lambda_k^{(1)})^2\lambda_k^{(2)}$	0	0	0	0	-3	0	0	0	6	0	0	-3	0
$\tilde{\lambda}(\lambda_k^{(1)})^2\zeta_2$	0	0	0	0	-1/2	1/2	0	0	1/2	-1/2	0	0	0
$\tilde{\lambda}\lambda_k^{(2)}\zeta_{1:2}$	0	0	0	0	-1/4	0	1/4	0	1/2	-1/2	0	0	0
$\tilde{\lambda}\zeta_{1:2}\zeta_2$	0	0	0	0	-1/8	1/4	1/8	-1/4	1/4	-3/4	1/2	0	0
$\tilde{\lambda}\lambda_k^{(1)}\zeta_{12}$	0	0	0	0	-1/2	0	1/2	0	1	-1	0	0	0
$\tilde{\lambda}\zeta_{1:22}$	0	0	0	0	-1/2	1/2	0	0	3/2	-3/2	0	-3/2	3/2
$\tilde{\lambda}^2(\lambda_k^{(2)})^2$	0	0	0	0	0	0	0	0	-1	0	0	1	0
$\tilde{\lambda}^2\lambda_k^{(2)}\zeta_2$	0	0	0	0	0	0	0	0	-1/2	1/2	0	0	0
$\tilde{\lambda}^2\zeta_2^2$	0	0	0	0	0	0	0	0	-1/8	3/8	-1/4	0	0
$\tilde{\lambda}^2\zeta_{2:2}$	0	0	0	0	0	0	0	0	-1/4	1/4	0	1/2	-1/2
$\tilde{\lambda}^2\lambda_k^{(1)}\lambda_k^{(3)}$	0	0	0	0	0	0	0	0	-2	0	0	2	0
$\tilde{\lambda}^2\lambda_k^{(1)}\zeta_3$	0	0	0	0	0	0	0	0	-1/2	1/2	0	0	0
$\tilde{\lambda}^2\zeta_{13}$	0	0	0	0	0	0	0	0	-1/2	1/2	0	1	-1

**Table 3** – Coefficients  $\gamma_{k_1 k_2 k_3 k_4 k_5}^{(4,m)}$ . We assume here  $\zeta_1 = 0$ .

Hereinafter, we shall consider perturbative series truncated at different orders, for which we find it worthwhile introducing the notation

$$\lambda_k^{(n)} \equiv \sum_{i=0}^n \lambda_k^{(i)}. \quad (4.1)$$

Fig. 2 shows an illustrative example of perturbative reconstructions at  $v = 4$ , which we shall focus on in this and next section. Plots have been produced as follows. First of all, in order to probe perturbation theory on an eigenvalue spectrum characterized by a relatively large condition number, we have chosen  $\lambda_{\text{ex}} = \{0.1, 0.3, 0.8, 2.2\}$  as the full eigenvalue spectrum to reconstruct. By means of numerical techniques detailed in [1], we have computed  $\mu = \tau_\rho \cdot \lambda_{\text{ex}}$  for several values of  $\rho$ . In correspondence with each pair  $(\rho, \mu)$  we have finally reconstructed the eigenvalue spectrum up to the fourth perturbative order, having chosen in all cases  $\tilde{\mu} = \bar{\mu}$ . We notice from the plots that the error made upon truncating the expansion at a given order increases in general at lower values of  $\rho$ . Moreover, the error is larger for eigenvalues at the edges of the spectrum and milder in the center of it. Remarkably, the convergence pattern of the lower half of the spectrum is radically different from the upper half. In the former case the perturbative series seems indeed to converge with alternate signs, whereas in the latter it displays an almost monotonic character.

In order to explain the observed behavior, let us first concentrate on the leading contribution  $\tilde{\lambda}$ . If  $(\rho^*, \tilde{\mu})$  fulfills the constraint  $0 \leq \tilde{\mu} \leq \rho^*(v+2)^{-1}$ , a solution  $\tilde{\lambda}$  to eq. (1.3) exists for all pairs  $(\rho, \tilde{\mu})$  with  $\rho \geq \rho^*$ . In this sense we can consider  $\tilde{\lambda}$  as a function of  $\rho$  at fixed  $\tilde{\mu}$ . The analytic form of  $\mathcal{T}_\rho$  tells us that  $\lim_{\rho \rightarrow \infty} \tilde{\lambda} = \tilde{\mu}$ . Now, we know for sure (modulo conjectures on the structure of  $\mathfrak{S}_B$ ) that  $\bar{\mu} \leq \rho^*(v+2)^{-1}$  provided  $\mu \in \mathcal{D}(\tau_\rho^{-1})$ . Since  $\mu_1 \leq \bar{\mu} \leq \mu_v$ , by continuity we conclude that

$$\exists \hat{\rho} : \quad \mathcal{T}_\rho^{-1}(\mu_1) \leq \tilde{\lambda} \leq \mathcal{T}_\rho^{-1}(\mu_v) \quad \forall \rho \geq \hat{\rho}. \quad (4.2)$$

At sufficiently large  $\rho$  the above inequality holds true of course with  $\mu_1$  and  $\mu_v$  respectively replaced by  $\mu_i$  and  $\mu_{i+1}$ , where  $i$  is such that  $\mu_i \leq \bar{\mu} \leq \mu_{i+1}$ . Eq. (4.2) does not tell where  $\tilde{\lambda}$  is placed in relation to the full eigenvalue spectrum. To find such an estimate, we can resort to eq. (2.12) of [1]. Based on arguments which are completely analogous to those used in there, we arrive easily at

$$\frac{\rho}{2v+1} \frac{M(v, v+3/2, \rho/(2\lambda_1))}{M(v, v+1/2, \rho/(2\lambda_1))} \leq \mathcal{T}_\rho(\tilde{\lambda}) \leq \frac{\rho}{3} \frac{M(1, 5/2, \rho/(2\lambda_v))}{M(1, 3/2, \rho/(2\lambda_v))}, \quad (4.3)$$

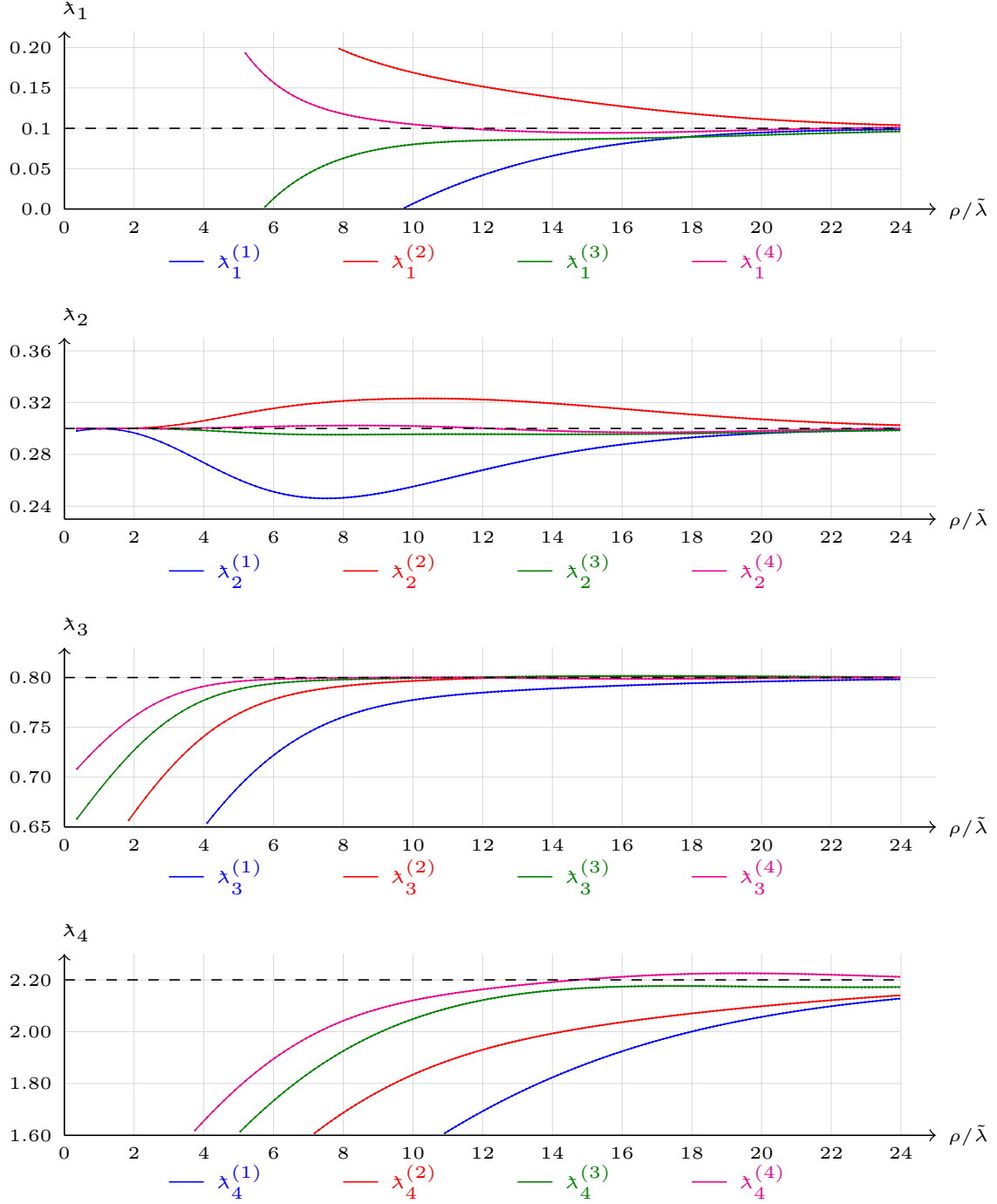
with  $M(a, b, z)$  denoting a Kummer function, *viz.*

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(a)_n}{(b)_n} z^n, \quad (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}. \quad (4.4)$$

As a consequence of the asymptotic limit  $\mathcal{T}_\rho(\tilde{\lambda}) \xrightarrow{\rho \rightarrow \infty} \tilde{\lambda}$  and (see *e.g.* chap. 13 of [7])

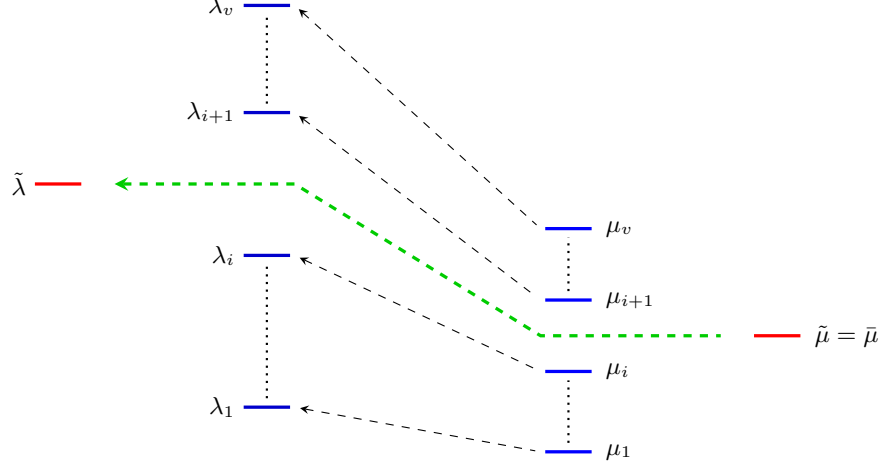
$$\frac{\rho}{2v+1} \frac{M(v, v+3/2, \rho/(2\lambda_1))}{M(v, v+1/2, \rho/(2\lambda_1))} \xrightarrow{\rho \rightarrow \infty} \lambda_1, \quad (4.5)$$

$$\frac{\rho}{3} \frac{M(1, 5/2, \rho/(2\lambda_v))}{M(1, 3/2, \rho/(2\lambda_v))} \xrightarrow{\rho \rightarrow \infty} \lambda_v, \quad (4.6)$$



**Fig. 2** – An example of perturbative eigenvalue reconstruction at  $v = 4$ . The unconstrained spectrum  $\lambda_{\text{ex}} = \{0.1, 0.3, 0.8, 2.2\}$  is represented by black dashed lines. The solid lines correspond to the four levels of approximation obtained by truncating the perturbative series at the first to fourth order.

we conclude that if  $\rho$  is sufficiently large, then  $\lambda_1 \leq \tilde{\lambda} \leq \lambda_v$ . As intuitively expected, non-linear effects are mitigated in the region of weak truncation. The situation is qualitatively depicted



**Fig. 3** – A schematic diagram showing the action of the operators  $\tau_\rho^{-1}$  (black dashed lines) and  $\mathcal{T}_\rho^{-1}$  (green thick dashed line) when  $\tilde{\mu} = \bar{\mu}$  and  $\mu \in \mathcal{D}(\tau_\rho^{-1})$ . If  $1 \leq i \leq v-1$  is such that  $\mu_i \leq \bar{\mu} \leq \mu_{i+1}$ , then at sufficiently large  $\rho$  we also have  $\lambda_i \leq \tilde{\lambda} \leq \lambda_{i+1}$ .

in Fig. 3.

Let us now consider the first few corrections to  $\tilde{\lambda}$ . As far as we are concerned with their numerical computation, we can limit ourselves to invert eqs. (3.18) one by one by means of a linear solver. On the other hand, the matrix structure of  $\mathcal{J}^{-1}$  (whose off-diagonal entries are all the same) allows us to perform the inversion in closed form. If the expression obtained as a result has too many contributions, we shall hardly find it better. Certainly, this is not the case with the first few perturbative corrections, of which we want to estimate the range of variation.

Since  $\mathcal{O}_{k;m}^{(n)}$  is either index-free or dependent upon  $k$  via monomials  $(\lambda_k^{(i_1)})^{m_1} \dots (\lambda_k^{(i_r)})^{m_r}$  with  $i_1 m_1 + \dots + i_r m_r \leq n$ , the only algebraic ingredients we need for the analytic inversion of eqs. (3.18) are the sums

$$i) \quad \sum_{k=1}^v (\mathcal{J}^{-1})_{jk} = 2\mathcal{D}^{-1}, \quad \mathcal{D} \equiv (v+2) \frac{F_{v+4}}{F_v} - v \frac{F_{v+2}^2}{F_v^2}, \quad (4.7)$$

$$ii) \quad \sum_{j=1}^v (\mathcal{J}^{-1})_{kj} (\lambda_j^{(i_1)})^{m_1} \dots (\lambda_j^{(i_r)})^{m_r} \\ = (\lambda_k^{(i_1)})^{m_1} \dots (\lambda_k^{(i_r)})^{m_r} \frac{F_v}{F_{v+4}} - \zeta_{i_1:m_1 \dots i_r:m_r} \mathcal{D}^{-1} \left( \frac{F_{v+4}}{F_v} - \frac{F_{v+2}^2}{F_v^2} \right), \quad (4.8)$$

where  $\zeta_{i_1:m_1 \dots i_r:m_r} \equiv \sum_{j=1}^v (\lambda_j^{(i_1)})^{m_1} \dots (\lambda_j^{(i_r)})^{m_r}$  is defined in perfect analogy with eqs. (3.23)–(3.25). We are thus ready to work out the algebra. In first place, a straightforward calculation yields

$$\lambda_k^{(1)} = \frac{F_v}{F_{v+4}} (\mu_k - \bar{\mu}) - 2\mathcal{D}^{-1} (\tilde{\mu} - \bar{\mu}), \quad (4.9)$$

whence we infer that

$$\text{if } \tilde{\mu} = \bar{\mu} \quad \Rightarrow \quad \text{sign}(\lambda_k^{(1)}) = \text{sign}(\mu_k - \bar{\mu}). \quad (4.10)$$

We thus recognize that the first perturbative correction to the leading term  $\tilde{\lambda}$  is positive for the upper part of the spectrum and negative for the lower one.

The algebraic evaluation of the second perturbative correction to  $\tilde{\lambda}$  is as easy as the first one, yet estimating its range of variation is somewhat more difficult. Let us set in this case  $\tilde{\mu} = \bar{\mu}$  from the beginning. Under this assumption, we see that the *r.h.s.* of eq. (3.38) reduces to a linear combination of four terms, namely

$$\mathcal{G}_k^{(2)} = \sum_{m=1}^4 \mathcal{G}_k^{(2,m)}, \quad (4.11)$$

$$\mathcal{G}_k^{(2,1)} = -\frac{1}{4\tilde{\lambda}} \left[ \zeta_{1:2} + 4(\lambda_k^{(1)})^2 \right] \frac{F_{v+6}}{F_v}, \quad \mathcal{G}_k^{(2,2)} = \frac{\zeta_{1:2}}{4\tilde{\lambda}} \frac{F_{v+4}F_{v+2}}{F_v^2}, \quad (4.12)$$

$$\mathcal{G}_k^{(2,3)} = \frac{1}{2\tilde{\lambda}} [\zeta_{1:2} + 2(\lambda_k^{(1)})^2] \frac{F_{v+4}}{F_v}, \quad \mathcal{G}_k^{(2,4)} = -\frac{\zeta_{1:2}}{2\tilde{\lambda}} \frac{F_{v+2}^2}{F_v^2}. \quad (4.13)$$

On inverting eq. (3.38) we find

$$\lambda_k^{(2)} = \sum_{m=1}^4 \lambda_k^{(2,m)}, \quad (4.14)$$

$$\lambda_k^{(2,1)} = -\frac{(\lambda_k^{(1)})^2}{\tilde{\lambda}} \frac{F_{v+6}}{F_{v+4}} + \frac{\zeta_{1:2}}{2\tilde{\lambda}} \frac{F_{v+6}}{F_{v+4}} \mathcal{D}^{-1} \left( \frac{F_{v+4}}{F_v} - 2 \frac{F_{v+2}^2}{F_v^2} \right), \quad (4.15)$$

$$\lambda_k^{(2,2)} = \frac{\zeta_{1:2}}{2\tilde{\lambda}} \mathcal{D}^{-1} \frac{F_{v+4}F_{v+2}}{F_v^2}, \quad (4.16)$$

$$\lambda_k^{(2,3)} = \frac{(\lambda_k^{(1)})^2}{\tilde{\lambda}} + \frac{\zeta_{1:2}}{\tilde{\lambda}} \mathcal{D}^{-1} \frac{F_{v+2}^2}{F_v^2}, \quad (4.17)$$

$$\lambda_k^{(2,4)} = -\frac{\zeta_{1:2}}{\tilde{\lambda}} \mathcal{D}^{-1} \frac{F_{v+2}^2}{F_v^2}. \quad (4.18)$$

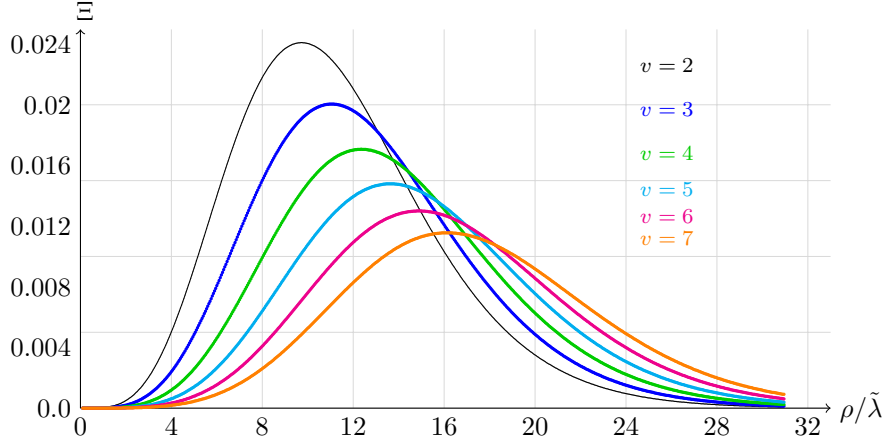
Hence, we have

$$\lambda_k^{(2)} = \frac{(\lambda_k^{(1)})^2}{\tilde{\lambda}} \left( 1 - \frac{F_{v+6}}{F_{v+4}} \right) + \frac{\zeta_{1:2}}{2\tilde{\lambda}} \mathcal{D}^{-1} \left( \frac{F_{v+6}}{F_v} + \frac{F_{v+4}F_{v+2}}{F_v^2} - 2 \frac{F_{v+6}}{F_{v+4}} \frac{F_{v+2}^2}{F_v^2} \right). \quad (4.19)$$

Since  $F_{v+6} < F_{v+4}$ , the first contribution to the *r.h.s.* is certainly positive. As for the second one, we define

$$\Xi = \frac{F_{v+6}}{F_v} + \frac{F_{v+4}F_{v+2}}{F_v^2} - 2 \frac{F_{v+6}}{F_{v+4}} \frac{F_{v+2}^2}{F_v^2}. \quad (4.20)$$

Plots reported in Fig. 4 suggest  $\Xi \geq 0$  and consequently  $\lambda_k^{(2)} \geq 0$ . Unfortunately, we lack at the moment an analytic proof of such inequality. Nevertheless, if the numerical evidence is correct, we can conclude that all the eigenvalues receive a positive contribution from the second perturbative correction, which together with eq. (4.10) explains qualitatively why the lower part of the spectrum seems to converge with alternate signs, whereas the upper part has an almost monotonic character.



**Fig. 4** – Positivity of the linear combination  $\Xi$  of  $\chi^2$ -ratios.

## 5 Perturbative estimators vs. the iterative one

In the last part of the paper we introduce a source of statistical uncertainty. So far we have studied the eigenvalue reconstruction under the hypothesis that  $\mu = \tau_\rho \cdot \lambda$  represents the exact truncated counterpart of some  $\lambda \in \mathbb{R}^v$ . This is rather unusual in most applications, where  $\mu$  is not the result of an exact truncation, but is instead estimated from a normal population  $\mathcal{P}_N = \{x^{(k)}\}_{k=1}^N$  of finite size  $N$ , distributed with  $\Sigma = \text{diag}(\lambda)$ . The sample estimate we consider here is performed as follows: a certain subset of  $M < N$  elements of  $\mathcal{P}_N$  falls within  $\mathcal{B}_v(\rho)$ , with the fraction  $M/N$  fulfilling  $\lim_{N \rightarrow \infty} M/N = \alpha(\rho; \lambda)$ . From this subset we measure  $\mathfrak{S}_{\mathcal{B}}$  via the classical estimator

$$(\hat{\mathfrak{S}}_{\mathcal{B}})_{ij} = \frac{1}{M-1} \sum_{k=1}^N (x^{(k)} - \bar{x})_i \cdot (x^{(k)} - \bar{x})_j \cdot \mathbb{I}[x^{(k)} \in \mathcal{B}_v(\rho)], \quad (5.1)$$

$$\bar{x}_i = \frac{1}{M} \sum_{k=1}^N x_i^{(k)} \cdot \mathbb{I}[x^{(k)} \in \mathcal{B}_v(\rho)]. \quad (5.2)$$

The eigenvalue spectrum  $\hat{\mu}$  of  $\hat{\mathfrak{S}}_{\mathcal{B}}$  represents our definition of the sample estimate of  $\mu$ , which we use as an input parameter for the reconstruction of  $\lambda$ . Of course,  $\hat{\mu}$  is interpreted as the realization of a stochastic variable in sample space. It thus makes sense to pose the question of what the statistical properties of the stochastic variable  $\hat{\lambda} = \tau_\rho^{-1} \cdot \hat{\mu}$  and its perturbative approximations  $\hat{\lambda}^{(k)}$  ( $k = 1, \dots, 4$ ) are.

Finding analytic relations between the expectation value in sample space of polynomial functions of  $\hat{\mu}$  and analogous functions of  $\hat{\lambda}$  is a difficult task, since we dispose of no analytic representation of the reconstruction operator  $\tau_\rho^{-1}$ . This goes beyond the aims of the present paper, so we limit ourselves to perform a simulative study in the specific case where  $\mathcal{P}_N$  distributes according to  $\Sigma = \text{diag}(\lambda_{\text{ex}})$ , with  $\lambda_{\text{ex}}$  introduced in the previous section. In our study, we have chosen  $N = 200, 250, \dots, 2000$ ; for each value of  $N$ , we have generated about 5000 normal populations; for each of them, we have then considered Euclidean balls with  $\rho = 4.0, 6.0, \dots, 40.0$  and for each pair  $(\rho, N)$  we have finally measured bias and variance of  $\hat{\lambda}$  and  $\hat{\lambda}^{(k)}$ . As an example, in Fig. 5 we



report our results at  $\rho = 6.0$  (corresponding to a weak truncation with  $\alpha(6.0; \lambda_{\text{ex}}) \simeq 0.844$ )<sup>5</sup>. From the plots on the left we notice that

- i) the bias of  $\hat{\lambda}^{(k)}$  is weakly sensitive to  $N$  for all  $k$ 's; it converges asymptotically to the intrinsic perturbative bias corresponding to  $\mu = \lim_{N \rightarrow \infty} \hat{\mu}$ , with finite size corrections proportional to  $1/N$ ;
- ii)  $\hat{\lambda}$  is slightly biased at finite  $N$  and asymptotically unbiased; convergence is again reached linearly in  $1/N$ .

Similarly, from the plots on the right we observe that

- iii) all variances vanish linearly in  $1/N$ ;
- iv) the variance of the higher eigenvalues increases at fixed  $N$  as we add perturbative corrections;
- v) by contrast, the variance of the lower eigenvalues decreases at fixed  $N$  as we add perturbative corrections;
- vi) the iterative estimator has a higher variance than all its perturbative approximations at the top of the spectrum and a lower one at the bottom of it.

The variance plots illustrate the potential usefulness of the perturbative estimators. The reconstruction of the upper part of the spectrum achieved from the iterative algorithm is rather noisy when  $N$  is moderately small (*e.g.* in this case  $N \lesssim 400$ ). The perturbative estimators allow to control the variance, the price to pay being the introduction of an asymptotic non-vanishing bias. Depending on the specific context, there will be an optimal choice for the order of the perturbative approximation, which guarantees acceptable values of both bias and variance.

Results are qualitatively similar for the other simulated values of  $\rho$ : the asymptotic biases and the slopes of the variances decrease as  $\rho$  increases, as naïvely expected.

A noticeable outcome of our simulations is inferred upon relating the variances of the reconstructed eigenvalues to those of the truncated ones. To this aim, we observe that since  $\lambda = \tau_\rho^{-1} \cdot \mu$  is a vector equation, each component of the reconstructed spectrum  $\lambda_i = \lambda_i(\mu_1, \dots, \mu_v)$  depends upon all the components of  $\mu$ . Hence, it follows that  $\text{var}(\lambda_i)$  is itself a function of all the components of  $\mu$ . Nevertheless, if eq. (3.8) holds true,  $\lambda_i$  depends weakly on  $\mu_k$  for  $k \neq i$ . Therefore, it makes sense to look at how  $\text{var}(\hat{\lambda}_i)$  relates to  $\text{var}(\hat{\mu}_i)$ . An example of such dependence is shown in Fig. 6 for  $i = 1, 4$ , corresponding respectively to the lowest and highest components of  $\lambda_{\text{ex}}$ . We first note that points displace along definite curves, in accordance with the conjecture of eq. (3.8). Then we note that the variances are linearly related, except for weak nonlinear effects observed at  $\text{var}(\hat{\mu}_4) \simeq 1.0 \times 10^{-2}$ . Again, we observe that the variance of the iterative estimator of the lowest eigenvalue is minimal and that of the highest one is maximal. What is most remarkable is the slopes observed for the highest eigenvalue  $\hat{\lambda}_4$ . On comparing the scales of the  $x$ - and  $y$ -axis, we recognize that a huge inflation of the variance occurs as a result of applying  $\tau_\rho^{-1}$  to  $\hat{\mu}$ . Numerical simulations

---

<sup>5</sup>statistical errors of the sample estimate of the variances have been computed according to the general formula for the standard error  $\text{se}(\text{var}(\hat{\lambda}_k)) = \widehat{\text{var}(\hat{\lambda}_k)} \cdot [2/(N-1) + \hat{\kappa}/N]$ , where  $\widehat{\text{var}(\hat{\lambda}_k)}$  is the sample estimate of  $\text{var}(\hat{\lambda}_k)$  and  $\hat{\kappa}$  is the sample excess kurtosis of the distribution of  $\text{var}(\hat{\lambda}_k)$ .

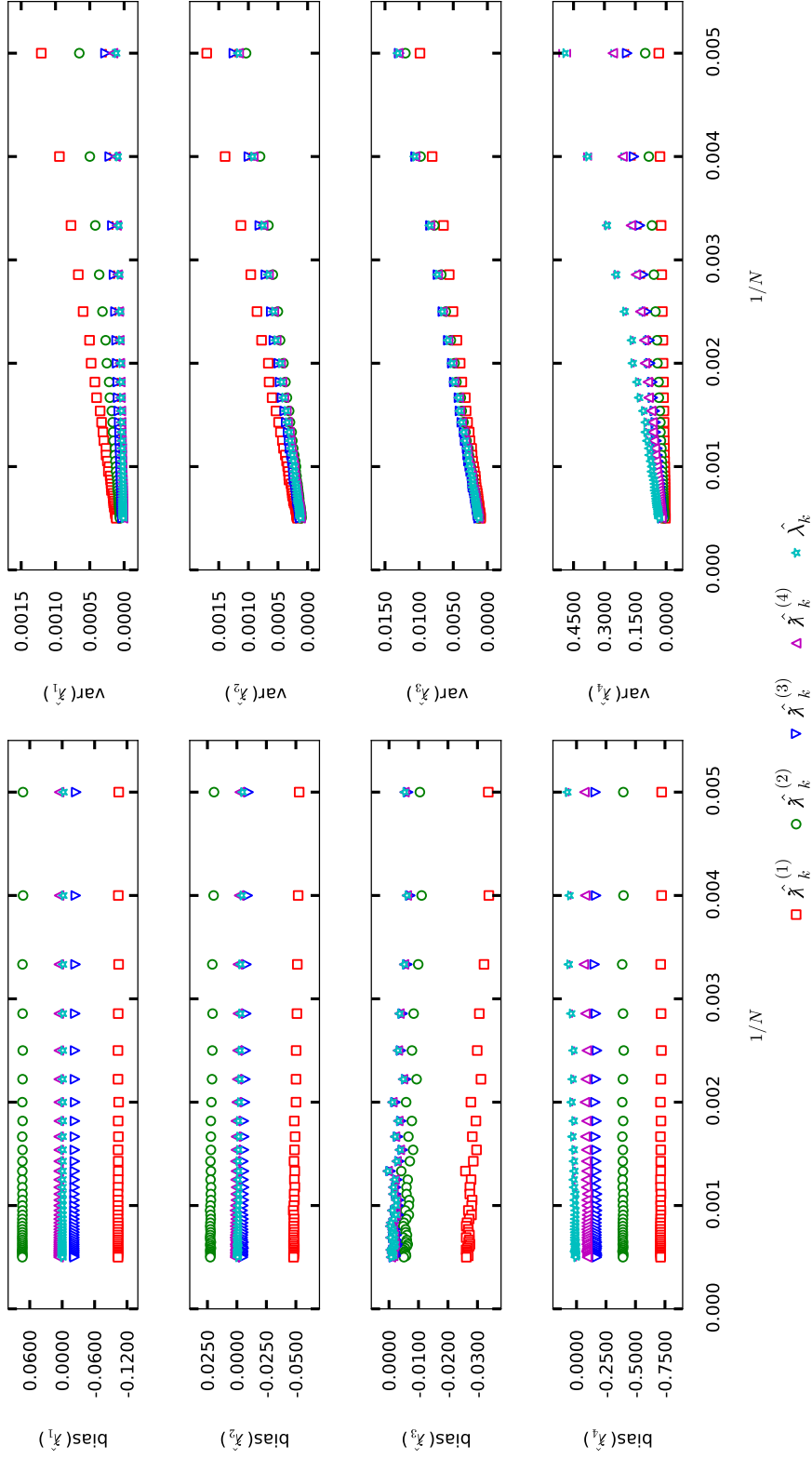
signal the existence of such amplification phenomena, yet they are not able to unveil the underlying mechanism. One should be anyway aware that in practical situations the exact reconstruction of the highest eigenvalue may be critical. In such cases, the adoption of perturbative estimators in place of the iterative one may represent a good way out.

## 6 Conclusions

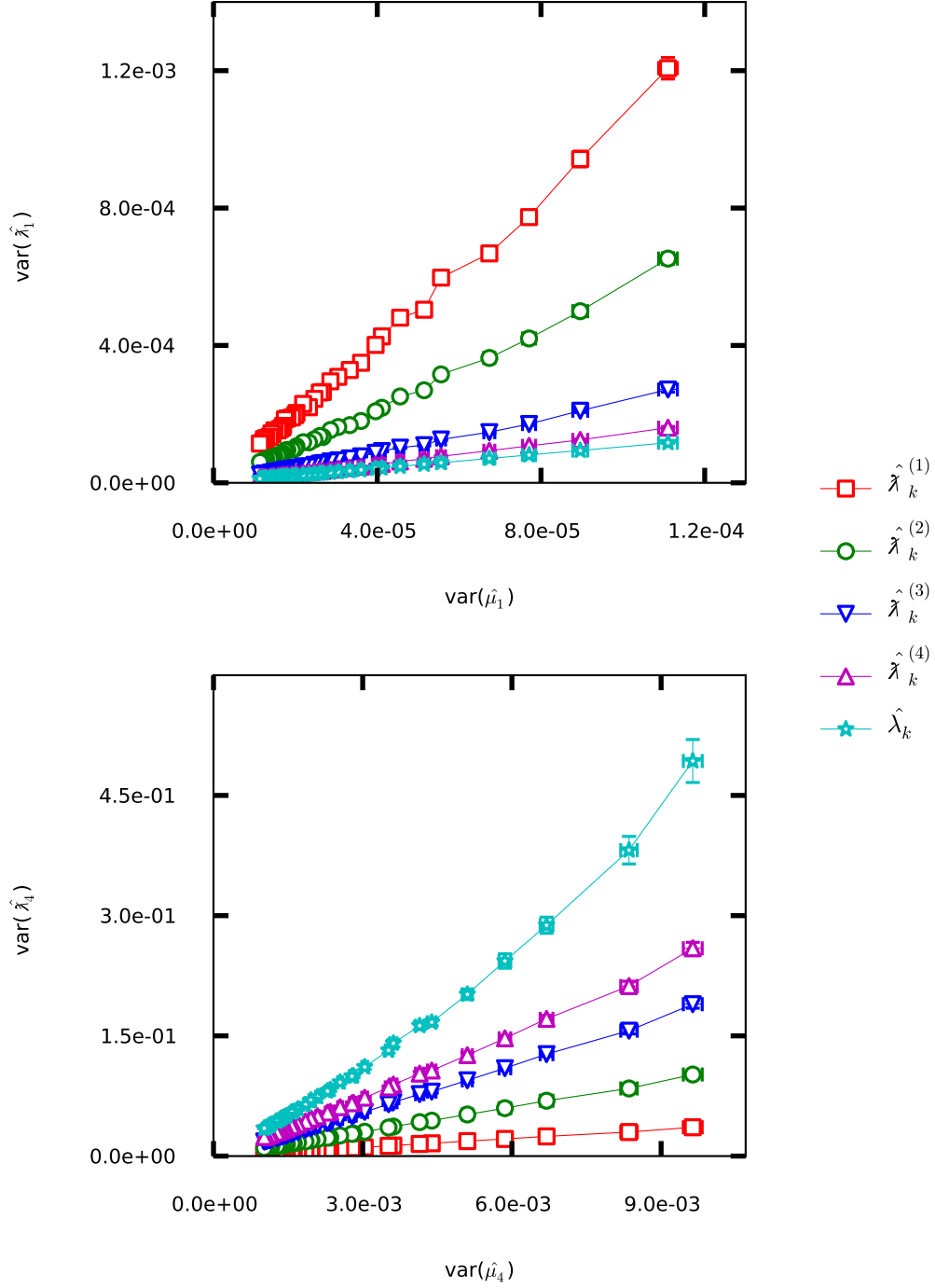
In this paper we have explored a perturbative approach to the reconstruction of a normal covariance matrix  $\Sigma$  from a spherically truncated counterpart  $\mathfrak{S}_B$ . We recall that  $\Sigma$  and  $\mathfrak{S}_B$  commute, so the reconstruction problem concerns only the eigenvalue spectra. Having preliminarily collected all the ingredients needed for the implementation of the perturbative expansion, we have detailed both its general features and the practical aspects related to the calculation of the perturbative coefficients. In the paper we provide formulae for the reconstruction of the eigenvalues of  $\Sigma$  up to the fourth perturbative order as well as a prototype MAPLE<sup>TM</sup> code to further improve the approximation.

From a theoretical point of view, the perturbative method is meant to complement the fixed-point iterative algorithm proposed by us in ref. [1] in cases where the latter becomes inefficient. Such cases occur when  $\rho$  is comparable or less than the lowest eigenvalue of  $\Sigma$  and/or the number of dimensions  $v$  is very large. In both cases, the inefficiency consists in a slow convergence speed. A second limit of the iterative algorithm shows up when the eigenvalue reconstruction is performed from statistically poor sample estimates of the truncated covariance spectrum. Specifically, we have shown that a statistical uncertainty on the truncated eigenvalues is inflated by the application of the reconstruction operator, thus producing large fluctuations of the higher components of the reconstructed spectrum. Perturbation theory offers the possibility to control the variance and stabilize the reconstruction by properly choosing the approximation order. The price to pay upon replacing the iterative estimator with its perturbative approximations is the introduction of an asymptotic non-vanishing bias. Of course, it is possible to adopt mixed approaches, where the lower part of the spectrum is reconstructed via the iterative estimator while the upper part is obtained from a perturbative one.

We conclude by recalling that both approaches developed in this paper and ref. [1] are based on some conjectured correlation inequalities over Euclidean balls, which have been extensively tested via numerical simulations and will be further analyzed elsewhere.



**Fig. 5** – Bias and variance of the iterative and perturbative estimators vs. the inverse of the population size at  $\rho = 6.0$ . Normal populations have been generated with  $\Sigma = \text{diag}(\lambda_{\text{ex}})$ .



**Fig. 6** – Variances of the iterative and perturbative estimators of the lowest and highest reconstructed eigenvalue vs. the variance of the corresponding truncated eigenvalue. Normal populations have been generated with  $\Sigma = \text{diag}(\lambda_{\text{ex}})$ .

## Appendix A MAPLE™ code

### A.1 Code block 1: the coefficient $\Delta_{i_1 \dots i_n}$

```
# Delta coefficient
# -----

Delta := proc()
  local SortArgs, V, ActCtr, NxtCtr, Res, k:
  for k from 1 to _npassed do
    if not type(_passed[k], 'nonnegint') then
      return 'procname(_passed)':
    end if:
  end do:
  SortArgs := sort([_passed[1.._npassed]]):
  V := Vector(_npassed):
  V[1] := 1:
  ActCtr := 1:
  NxtCtr := 2:
  for k from 1 to (_npassed-1) do
    if SortArgs[NxtCtr] = SortArgs[ActCtr] then
      V[ActCtr] := V[ActCtr]+1:
    else
      ActCtr := NxtCtr:
      V[ActCtr] := 1:
    end if:
    NxtCtr := NxtCtr+1:
  end do:
  Res := 1:
  for k from 1 to _npassed do
    Res := Res*(doublefactorial(2*V[k]-1)):
  end do:
  return Res:
end:
```

### A.2 Code block 2: perturbative expansion of eq. (1.1)

```
# Nested sequence
# -----

NestSeq := proc(TheEq, v::nonnegint, niter::nonnegint)
  if niter = 0 then
    eval(TheEq):
  else
    seq(eval(NestSeq(TheEq, v, niter-1)),
        cat('r', niter) = 1..v):
  end if
end proc:

# Derivatives of Gaussian Integrals
# -----

DerAlpha := proc()
```

```

global v,Delta:
local m,Fact1,Fact2:
m := _npassed:
Fact1 := Delta(_passed[1.._npassed])/(2*1[0])^m:
Fact2 := add((-1)^(m-j)*binomial(m,j)*F[v+2*j],j=0..m):
return Fact1*Fact2:
end proc:

DerAlphak := proc()
global v,Delta:
local m,Fact1,Fact2:
m := _npassed-1:
Fact1 := Delta(_passed[1.._npassed])/(2*1[0])^m:
Fact2 := add((-1)^(m-j)*binomial(m,j)*F[v+2*(j+1)],j=0..m):
return Fact1*Fact2:
end proc:

# Function arguments
# -----

lam := Vector(v):
for k from 1 to v do
  lam[k] := add(l[j,k]*epsilon^j,j=0..n):
end do:
lam := seq(lam[k],k=1..v):
lam0 := seq(l[0,k],k=1..v):

# Integral ratio
# -----

R := proc(j)
global lam:
return alpha[j](lam)/alpha(lam):
end:

# Taylor expansion of the map
# -----

for j from 1 to n do
  for k from 1 to v do
    Rk := convert(taylor(R(k),epsilon=0,n+1),polynom):
    h[j,k] := expand(coeff(lam[k]*Rk,epsilon,j)):
  end do:
end do:

# Evaluation conditions
# -----

COA := alpha(lam0)=F[v]:
COAk := seq(alpha[k](lam0)=F[v+2],k=1..v):

C1A := NestSeq(D[r1](alpha)(lam0)=DerAlpha(r1),v,1):
C1Ak := NestSeq(D[r1](alpha[r2])(lam0)=DerAlphak(r1,r2),v,2):

C2A := NestSeq(D[r1,r2](alpha)(lam0)=DerAlpha(r1,r2),v,2):
C2Ak := NestSeq(D[r1,r2](alpha[r3])(lam0)=DerAlphak(r1,r2,r3),v,3):

```

```

C3A := NestSeq(D[r1,r2,r3](alpha)(lam0)=DerAlpha(r1,r2,r3),v,3):
C3Ak := NestSeq(D[r1,r2,r3](alpha[r4])(lam0)=DerAlphak(r1,r2,r3,r4),v,4):

C4A := NestSeq(D[r1,r2,r3,r4](alpha)(lam0)=DerAlpha(r1,r2,r3,r4),v,4):
C4Ak := NestSeq(D[r1,r2,r3,r4](alpha[r5])(lam0)=DerAlphak(r1,r2,r3,r4,r5),v,5):

CArg0 := seq(l[0,k]=l[0],k=1..v):

# Evaluations
# -----

for j from 1 to n do
  for k from 1 to v do
    h00[j,k] := expand(eval(h[j,k],[C1A])):
    h01[j,k] := expand(eval(h00[j,k],[C2A])):
    h02[j,k] := expand(eval(h01[j,k],[C3A])):
    h03[j,k] := expand(eval(h02[j,k],[C4A])):
    h04[j,k] := expand(eval(h03[j,k],[C1Ak])):
    h05[j,k] := expand(eval(h04[j,k],[C2Ak])):
    h06[j,k] := expand(eval(h05[j,k],[C3Ak])):
    h07[j,k] := expand(eval(h06[j,k],[C4Ak])):
    h08[j,k] := expand(eval(h07[j,k],[COA])):
    h09[j,k] := expand(eval(h08[j,k],[COAk])):
    h10[j,k] := expand(eval(h09[j,k],[CArg0])):
  end do:
end do:

```

### A.3 Code block 3: extraction of $\gamma_{k_1 \dots k_{n+1}}^{(n,m)}$

```

with(LinearAlgebra):
with(RandomTools):

v := 6:

# 0-structure matrix
# -----

zeta1 := add(l[1,k],k=1..v):
zeta2 := add(l[2,k],k=1..v):
zeta11 := add(l[1,k]^2,k=1..v):
zeta12 := add(l[1,k]*l[2,k],k=1..v):
zeta111 := add(l[1,k]^3,k=1..v):

S3matrix := Matrix(v,v):

for k from 1 to v do
  S3matrix[k,1] := l[1,k]^3:
  S3matrix[k,2] := expand(l[1,k]*zeta11):
  S3matrix[k,3] := expand(zeta111):
  S3matrix[k,4] := l[0]*l[1,k]*l[2,k]:
  S3matrix[k,5] := expand(l[0]*l[1,k]*zeta2):
  S3matrix[k,6] := expand(l[0]*zeta12):
end do:

```

```

# Jacobian matrix
# -----

Jmatrix := Matrix(v,v):

for k1 from 1 to v do
  for k2 from 1 to v do
    Jmatrix[k1,k2] := (1/2)*(Delta(k1,k2)*F[v+4]/F[v] - F[v+2]^2/F[v]^2):
  end do:
end do:

# Terms to be removed by hand
# -----

V := Vector(v):

for j from 1 to v do
  V[j] := 0:
  for k from 1 to v do:
    V[j] := V[j] + Jmatrix[j,k]*l[3,k]:
  end do:
end do:

# Randomized Linear system
# -----

C := Vector(v):

for j from 1 to v do
  lincond0 := l[0]=Generate(float(range=0..1,'method=uniform')):
  linvals1 := seq(Generate(float(range=0..1,'method=uniform')),m=1..v-1):
  lincond1 := seq(l[1,m]=linvals1[m],m=1..v-1):
  lincond1 := lincond1,l[1,v]=-add(linvals1[m],m=1..v-1):
  lincond2 := seq(l[2,m]=Generate(float(range = 0..1,'method=uniform')),m=1..v):
  for m from 1 to v do
    S3matrix[j,m] := eval(S3matrix[j,m],[lincond0,lincond1,lincond2]):
  end do:

  r := expand(h10[3,j] - V[j]):
  s := expand(eval(coeff(r,F[v+6]),F[v+2]=0)):
  C[j] := eval((l[0]^2)*F[v]*s,[lincond0,lincond1,lincond2]):
end do:

# (-1) x Gamma coefficients
# -----

Gcoefs := LinearSolve(S3matrix,C):

```



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